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The Tightness of the Teeth, Considered as a Problem Concerning the Equilibrium of a Thin Incompressible Elastic Membrane

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XII. *The Tightness of the Teeth, considered as a Problem concerning the Equilibrium of a Thin Incompressible Elastic Membrane.*

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Contents.

§ 1. *Introduction.*

A tooth may be regarded as a rigid body, held in a rigid socket by a thin membrane—the *periodontal membrane* or *pericementum*—which fills the space between the tooth and the bone, and is attached to each. This membrane, whose average thickness*

* *Cf.* Orban, 'Dental Histology and Embryology,' p. 127 (1928).

is only 0·23 to 0·25 mm. (0·0091 to 0·0099 in.), is called upon to supply the tractions necessary to resist the forces applied in biting or chewing.

An interesting problem in the theory of elasticity is thus suggested, namely, the determination of the displacement of the tooth and the strain and stress in the membrane, corresponding to the application of assigned forces to the crown of the tooth. We are obviously entitled to treat the problem mathematically as that of an infinitely thin membrane, but we shall have to make other simplifications in order to reduce the problem to a manageable form.

With this object in view, we shall first make two hypotheses, of which the former is reasonably likely to be true, but the latter much less so. These hypotheses are that the membrane is elastically (i) *homogeneous ;* (ii) *isotropic.*

Our next hypothesis is that the membrane is *incompressible.** We may justify this assumption from the fact that the tissue is largely composed of water, and that the only changes in volume arise when variations in pressure draw in extra blood from the circulatory system or squeeze it out; but these changes will probably be small unless the membrane is in a congested condition. Congestion, with consequent compressibility, might be expected theoretically to cause a slight loosening of the tooth. There is, however, another way of approaching this question of incompressibility, which is of interest. It is improbable that, in the normal functioning of the tooth, the membrane experiences finite† strain, or, in other words, that the linear elements of the membrane experience finite elongations. On the other hand, in order to balance finite applied forces, the stress in the membrane must be finite. Now finite stress and infinitesimal strain can coexist only if at least one of the elastic moduli is infinite. Having already assumed that the membrane is elastically isotropic, we have at our disposal two moduli, the modulus of compression and the rigidity. It is improbable that the rigidity is infinite; thus we are led to the assumption that the membrane is incompressible.

The elastic characteristics assumed above differ considerably from the structure sometimes assumed in dental research. The presence of fibres in the membrane, running across it from tooth to bone, has led to the idea that the membrane functions as a system of elastic cords.‡ Since, however, these fibres are usually not straight when seen in a microscopic section, it is difficult to see how they alone could account for the tightness possessed by a tooth. It is more probable that their effect on the elastic specification is to render the membrane anisotropic, although still incompressible. We shall not, however, take such anisotropy into consideration.

* *Cf.* GYSI, 'Dent. Digest,' vol. 36, p. 623 (1930).

† We are here only seeking basic hypotheses. The word "finite" is used in a loose sense. Physically, "infinitesimal" means "very small," "infinite" means "very large." If the strain were finite, the investigation would lie outside the scope of ordinary elastic theory.

‡ *Cf.* SCHWARZ, 'Dent. Items,' February (1930).

To sum up, our problem is as follows :—

An elastic membrane fills the space between, and is attached to, two rigid bodies, the tooth and the socket, of which the former is subjected to assigned forces, while the latter is held fixed. We assign to the membrane the following properties :—

(i) *infinitesimal, but not necessarily uniform, thickness ;*

(ii) *homogeneity ;*

(iii) *isotropy ;*

(iv) *incompressibility ;*

(v) *finite rigidity ;*

and we shall suppose that the free edge, or margin, of the membrane is subject to atmospheric pressure. The problem is to investigate the equilibrium of the system, particularly with regard to the displacement of the tooth and the strain and stress in the membrane.

As far as the differential equations are concerned, the problem is closely analogous to the problem of the motion of a thin layer of viscous liquid, as treated in the theory of lubrication. The analogy led A. G. M. Michell to suggest the experimental investigation of problems of lubrication by statical experiments on incompressible elastic media.*

Before proceeding to describe the contents of the paper, it may be well to state in a word what it is that accounts for the tightness of a tooth, held by a membrane of finite rigidity. It is the *pressure*, which, in the incompressible medium, takes the place of three of the ordinary components of stress. As we pass over the surface of the root, the pressure undergoes finite variations and these variations yield a finite resultant, capable of balancing finite applied forces. We find, in the course of the investigation, that the displacement of the tooth, as a result of the application of finite forces, is very small indeed, varying as the cube of the thickness of the membrane. This fact gives additional weight to the present theory, as explaining the tightness of the teeth, for, if the membrane were a compressible elastic body, we would find the displacement of the tooth directly proportional to the thickness of the membrane. Moreover, the present theory assigns a rapid increase in looseness with increase in the thickness of the membrane. Were the membrane to function as a system of elastic cords, the angular displacement of the tooth would be proportional to the thickness of the membrane (the load being supposed constant) : no matter how wide the membrane

* The following references to the hydrodynamical theory may be quoted : Reynolds, 'Phil. Trans.,' vol. 177, p. 157 (1886) ; 'Papers on Mechanical and Physical Subjects,' vol. 2, p. 228 ; Rayleigh, 'Phil. Mag.' (5), vol. 36, p. 354 (1893) ; 'Sci. Pap.,' vol. 4, p. 78 ; 'Phil. Mag.,' vol. 35, p. 1 (1918) ; 'Sci. Pap.,' vol. 6, p. 523 ; Sommerfeld, 'Z. Math. Phys.,' vol. 50, p. 97 (1904) ; Michell, *ibid.*, vol 52, p. 123 (1905). The analogy to the theory of the Prandtl boundary-layer (where the viscosity is small) is much less close ; for the application of tensor methods to this latter problem in its most general form, we may, however, refer to T. Levi-Civita, 'Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete,' p. 30, Aachen (1929).

became, it would always suffer the same fractional extension and compression at the margin, and there is no reason to suppose that the tooth would ultimately be forced against the bone. But since, on the basis of the present theory, the displacement varies as the cube of the thickness of the membrane, then, as the thickness of the membrane is increased, the tooth, displaced under constant load, continually approaches the bone. Thus if, with a membrane of normal thickness, the loading of the tooth reduced the thickness of the membrane at a point on the margin by one-tenth, then, with a membrane of twice the thickness, the same load would reduce the thickness by two-fifths, and, with a membrane of three times the thickness, the same load would reduce the thickness by nine-tenths, and so on; with a slightly thicker membrane, the effect of loading would be to force the tooth against the bone.

The component of the displacement of a point in the membrane, in a direction parallel to the surface of the root, is of the order of the square of the thickness of the membrane, while the component normal to the surface of the root is of the order of the cube of the thickness. The components of strain are of the order of the thickness of the membrane, so that the strain is a "small strain."

The subject of the paper was suggested to me by Professor H. K. Box, the problem having arisen in connection with his researches into the mechanical causes of diseases of the periodontal membrane. I am also indebted to him for valuable information and advice on the dental aspects of the problem. One of the principal objects of the investigation was to determine the position of the centre of rotation of a tooth under the action of a force applied to the crown. This question, which has interested research workers in dentistry, is answered in §§ 11 and 14 in respect of certain model teeth having membranes of constant thickness (a simplification unfortunately rendered necessary for mathematical reasons).

But perhaps a more important deduction in the present paper (involved qualitatively in the basic hypothesis of incompressibility) is the considerable local variation in pressure in the membrane when the tooth is subjected to the forces arising in occlusion. This has not, I think, been taken into consideration previously. I am not competent to discuss the physiological consequences of such local variations in pressure, and shall merely emphasise the fact that the application of a force of considerably less than one pound weight may reduce the pressure to zero at some points in the membrane, while raising it to two atmospheres at other points. In deducing numerical results in this connection, the only data required are geometrical measurements of teeth; the rigidity drops out of the equations, as also does the thickness of the membrane. The weakest point in the argument is the assumption of uniform thickness of the membrane, but we can at least hope to have obtained results of a rough quantitative significance.

Throughout the paper, the argument is kept as general as possible, and numerical values are only inserted in order to obtain final numerical results. While the arguments of Part I are very general, and would apply (as far as they go) to a multiple-rooted

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tooth, we shall concentrate our attention for final numerical results on the upper central tooth, for which we shall employ the following average measurements :—*

Length over all	0·88 inches.
Crown length	0·39 ,,
Root length	0·49 ,,
Mesio-distal diameter at neck (side to side)	0·24 ,,
Labio-lingual diameter (front to back)	0·27 ,,

Synopsis.

The first part of the paper (§§ 2–7) deals with the general problem of the equilibrium of a thin homogeneous isotropic incompressible elastic membrane, confined between and attached to, two rigid bodies, one of which is given a prescribed displacement. The results are mathematically the same as those first given by REYNOLDS (*loc. cit. supra*) in connection with the theory of lubrication, but an attempt has been made to present the argument in a more rigorous form than has hitherto been done, with reference, of course, to the elastic problem, rather than to the hydrodynamical. It is pointed out how the theory of elasticity is generally concerned with limiting solutions, or principal parts of solutions, of a system of partial differential equations containing a parameter. The manner in which the parameter enters into the present problem is different from that in which it enters into ordinary problems of elasticity, in that, in the present investigation, the range of the equations depends on the parameter. By a special choice of co-ordinates, this range is made independent of the parameter. Tensor notation is employed, and the principal results are contained in equations (56) to (59).

The problem of the tooth is actually a three-dimensional problem, but as the analogous two-dimensional problem presents some interesting and simple features, it receives attention in Part II (§§ 8–11). The determination of the pressure in the membrane in terms of the displacement of the tooth is reduced to quadratures in (71) and the displacement of the tooth is connected with the applied force-system in (80). In § 9 the positions of the points of maximum and minimum pressure are investigated, and it is found that these points are in general arranged on a circle, having for centre the centre of rotation of the tooth, and for radius the radius of gyration of a fictitious wire, coincident with the longitudinal section of the root, with a linear density equal to the inverse cube of the thickness of the membrane. The results are applied, in § 11, to a wedge-shaped model of the upper central tooth. It is found that the centre of rotation lies near the middle point of the root, on the side away from the apex (see (89)). The "critical" axial load which, when applied as a pulling force, reduces the pressure at the apex to zero, and when applied as a pressure, raises the pressure at the apex to two atmospheres, is found to be approximately 0·65 lbs. The critical transverse load (*i.e.*, that which reduces the pressure to zero at the point of lowest pressure) is approximately 0·45 lbs. The points of maximum and minimum pressure are located, and a graph shows the distribution of pressure under transverse load.

* These figures are taken from G. V. BLACK, 'Dental Anatomy,' Philadelphia, p. 19 (1902).

Part III (§§ 12–14) deals with the case of a root of revolution. The argument is, for the most part, confined to the case where the membrane is of uniform thickness. The determination of the pressure in the membrane is again reduced to quadratures. § 13 deals with the symmetrical case where the tooth receives a two-dimensional displacement; the loading corresponding to a given displacement is investigated. § 14 treats the case of a model of the upper central tooth whose root is a right circular cone; this is not a bad approximation to reality. The centre of rotation is found to lie in the half of the root farthest from the apex (see (142)). The critical axial load (defined as above) is found to be approximately 0·38 lbs., while the critical transverse load is approximately 0·19 lbs. (The critical load is in every case directly proportional to the atmospheric pressure, to which we have assigned, in making the numerical calculations, the value 15 lbs. per sq. in.) The lines of constant pressure on the root are shown in fig. 14.

Finally, in order to get some idea of the magnitude of the displacement of the tooth corresponding to an assigned load, there being no data available as to the rigidity of the periodontal membrane, we make the assumption that it has the rigidity of rubber. This leads to some remarkable results. The critical axial load of 0·38 lbs., applied either as a pull or a push, causes an axial displacement of only $2\cdot 8 \times 10^{-7}$ in., while the critical transverse load of 0·19 lbs., applied at the biting edge, gives to that edge a displacement of $8\cdot 9 \times 10^{-6}$ in. Even though the rigidity were considerably less than that of rubber (as it well may be), the above displacements are so extremely minute, that it may be claimed for the present theory that it provides an adequate explanation of the tightness of the teeth.

Part I.—General Theory.

§ 2. *Notation and equations of equilibrium of an incompressible homogeneous isotropic elastic medium.*

Let us refer the points of an elastic medium to any curvilinear co-ordinate system x^i $(i = 0, 1, 2)$. We shall give to italic indices the range 0, 1, 2, and to Greek indices the range 1, 2, with summations through these ranges in the case of a repeated index, in accordance with the usual convention.

The line-element will be of the form

$$ds^2 = a_{ij}\, dx^i\, dx^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall denote by A_i the operation of taking a covariant derivative with respect to the tensor a_{ij}, and write

$$A^i = a^{ij} A_j,$$

where a^{ij} is the co-factor of a_{ij} in the determinant $|a_{ij}|$, divided by that determinant. Then for an invariant V we have

$$A_i V = \frac{\partial V}{\partial x^i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for a vector V^i or V_i $(= a_{ij} V^j)$,

$$A_j V^i = \frac{\partial V^i}{\partial x^j} + F^i_{jk} V^k, \quad A_j V_i = a_{ik} A_j V^k = \frac{\partial V_i}{\partial x^j} - F^k_{ij} V_k, \qquad (3)$$

where F^i_{jk} is the Christoffel symbol of the second kind,

$$F^i_{jk} = \tfrac{1}{2} a^{il} \left(\frac{\partial a_{jl}}{\partial x^k} + \frac{\partial a_{kl}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l} \right).$$

For a mixed tensor V^i_k we have

$$A_j V^i_k = \frac{\partial V^i_k}{\partial x^j} + F^i_{jl} V^l_k - F^l_{jk} V^i_l. \qquad (4)$$

Denoting by u^i the (infinitesimal) components of displacement, we have for the covariant components of strain*

$$e_{ij} = \tfrac{1}{2} (A_i u_j + A_j u_i), \qquad (5)$$

and if, as we shall assume henceforth, the medium is *incompressible*, we have the relation

$$e_i^{\,i} \equiv A_i u^i = 0. \qquad (6)$$

Assuming now that the medium is *homogeneous* and *isotropic*, and denoting the covariant stress tensor by T_{ij}, we have the stress-strain relations

$$T_{ij} = -p a_{ij} + 2 \mu e_{ij}, \qquad (7)$$

where p is the pressure in the medium and μ the rigidity, a constant. We shall assume that there are no body-forces (we neglect gravity): the equations of equilibrium are then

$$A_j T_i^{\,j} = 0, \qquad (8)$$

which, on making use of (6) and the fact that $A_i A_j = A_j A_i$ (since the space is Euclidean), reduce to

$$A^i p = \mu A^j A_j u^i, \qquad (A^i p = a^{ij}\, \partial p / \partial x^j.) \qquad (9)$$

It follows immediately that p satisfies the harmonic equation

$$A^i A_i p = 0, \qquad (10)$$

and that the components of displacement satisfy the biharmonic equations

$$A^j A_j A^k A_k u^i = 0. \qquad (11)$$

The fundamental equations (6), (9), and the deductions (10), (11), are familiar in the theory of slow steady motions of viscous liquids.

* For a more detailed account of the application of tensor notation to elasticity, see, for example, APPELL, 'Mécanique Rationnelle,' vol. 5, p. 91 (1926).

§ 3. *Elastic problems as parametric problems.*

The argument just given, in which we spoke of "infinitesimal" components of displacement, has the degree of rigour customary in arguments on the theory of elasticity, which has not received the same critical attention as has been accorded, for example, to potential theory. The problem of the "thin" membrane, however, forces us to take a somewhat deeper view, and to realize more accurately what we are really doing in problems of elasticity. It is true that the general equations which we develop might be found in a few lines by following the intuitive arguments of REYNOLDS (*loc. cit.*) in connection with the theory of lubrication, but to adopt this point of view would be to lose sight of some of the most interesting aspects of the problem.

If we suppose that an elastic medium is subjected to given finite surface displacements, or given finite surface tractions, and if we suppose that the elastic moduli are finite, the determination of the displacement in the medium will be difficult and of doubtful physical validity without well-defined relations between stress and strain for the case where the strain is finite. What we actually do in most elastic problems is to introduce implicitly a parameter ε; we assume that the elastic moduli are functions of ε, tending to infinity as ε tends to zero, and that the surface displacements (when assigned) are also functions of ε, tending to zero with ε. We solve the differential equations for the components of displacement as if they were valid for finite displacements; the solutions will be functions of ε, and the components of stress (as given by the stress-strain relations) will also be functions of ε. The solution of the elastic problem is furnished by taking, for the components of stress, the limits of the solutions as ε tends to zero, and for the "infinitesimal" components of displacement, the principal parts (as ε tends to zero) of the solutions of the differential equations.

The above process, valid for most elastic problems, is not applicable to our problem of the thin membrane, because we are not going to suppose that the rigidity tends to infinity. (One of the moduli, it is true, has been made infinite in making the medium incompressible, but that is a limiting process that is over and done with.) The parameter does not enter our equations through the elastic moduli, but through the thickness of the membrane, which, we shall suppose, tends to zero with ε. In this case, then, we have quite a different situation from that which obtains in ordinary elastic problems. The parameter ε does not at first appear explicitly in our differential equations (since we do not assume that the rigidity depends on ε), but the range of validity of the differential equations depends on ε, and the surface displacements also depend on ε. By introducing a new system of co-ordinates, whose range is independent of ε, we bring our problem into line with ordinary problems in elasticity; the parameter ε, after the transformation of co-ordinates, appears explicitly in the differential equations.

§ 4. *The problem of the thin membrane.*

Let S and T be two rigid bodies; the same letters may, without ambiguity, be used to denote their surfaces. Let the region between them be filled (wholly or partially) with an incompressible elastic membrane of finite rigidity μ, the membrane being attached to both surfaces at every point where it comes into contact with them. There are two leading cases, shown in figs. 1*a* and 1*b*. In the former, the membrane forms a closed shell; in the latter it is terminated at a curve C drawn on T. More complicated circumstances may be considered under this latter case, the membrane being multiply-connected on account of holes. The simple case of fig. 1*b* is that of the tooth, and with it we are primarily concerned, but we can, up to a certain point, discuss all such cases on a common basis.

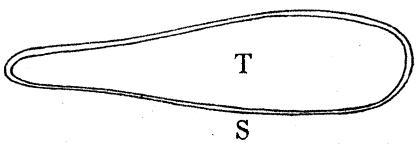

FIG. 1*a*.

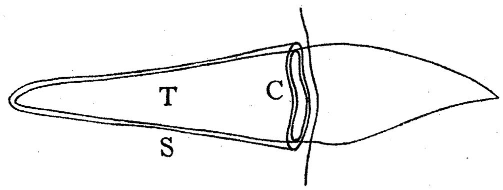

FIG. 1*b*.

The problem is as follows. The socket S is held fixed, and the tooth T is subjected to finite external forces. It is required to investigate the stress, strain, and displacement in the elastic membrane, assuming that its thickness is small. It will be assumed that the components of strain are small, and this assumption will be justified by showing that a small strain is capable of giving equilibrium when finite forces are applied to T.

If the components of strain are small, then by (7), the components of stress are given approximately by

$$T_{ij} = -pa_{ij}; \qquad (12)$$

in other words, the principal part of the stress consists of the pressure alone.* Since the forces applied to the tooth must be balanced by this pressure, it follows that the pressure p must undergo finite variations as we pass over the surface of the root of the tooth.

Analytically, our problem may be roughly described as follows. We have to find solutions p, u^i of the equations (6) and (9), such that $A_j u^i$ are small, while the tangential component of the pressure gradient is finite. As to boundary conditions, the displacement on S is zero; the displacement on T is a rigid body displacement, whose order of magnitude will be determined in the course of the argument, and which we may regard tentatively as prescribed. In the case of fig. 1*b*, we shall impose the further boundary condition $p = P$ on the normal section of the membrane at the bounding curve C, where P is a given constant (the atmospheric pressure). If this problem can be solved analytically, we can determine the force-couple resultant of the elastic forces exerted by the membrane on the tooth, corresponding to a prescribed rigid body displacement

* But the principal part of the stress-gradient is not the same as the gradient of the principal part of the pressure, since, although the first derivatives of the displacement in the membrane are infinitesimal, the second derivatives are not infinitesimal; *cf.* (56).

of the tooth. We can then invert the solution, and find the displacement of the tooth corresponding to prescribed external forces applied to the tooth. The solution will then be complete.

§ 5. *Normal co-ordinate system. Transformation of equations.*

We have to consider a singly infinite system of pictures of the unloaded equilibrium state as the thickness of the membrane is made to tend to zero. Let us decide to regard the surface S as fixed once for all, and on it let us choose a general curvilinear co-ordinate system x^α ($\alpha = 1, 2$). Let P be any point in the membrane and PN the perpendicular let fall from P on S. Let us write $x^0 = $ PN. Then x^i ($i = 0, 1, 2$) form a system of curvilinear co-ordinates in space, which will be regular provided that PN is less than the smaller of the two principal radii of curvature of S at N. Since we have only to deal with a region adjacent to S, we may assume this condition to be satisfied.

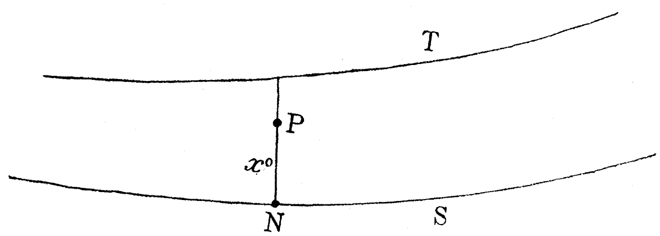

FIG. 2.

Let us write the equation of the surface T in the form

$$x^0 = \varepsilon\tau\,(x^1, x^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where τ is a regular function over S, and ε is a constant by which we control the thickness of the membrane ; it is the parameter of the elastic problem and will be made to tend to zero.

Now let P be any point of the membrane with co-ordinates x^i ; let us write

$$\xi = x^0/\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Then throughout the membrane ξ has the range 0 to $\tau\,(x^1, x^2)$.

Let us now transform the equations (6) and (9) by changing the independent variables from x^0, x^1, x^2 to ξ, x^1, x^2.

From the nature of the co-ordinate system, x^0, x^1, x^2, we have

$$a_{\alpha 0} = 0, \qquad a_{00} = 1, \qquad a^{\alpha 0} = 0, \qquad a^{00} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Hence, for the Christoffel symbols, we have

$$\Gamma^\alpha_{00} = \Gamma^0_{\alpha 0} = \Gamma^0_{00} = 0, \qquad \Gamma^0_{\alpha\beta} = -\tfrac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^0}, \qquad \Gamma^\alpha_{\beta 0} = \tfrac{1}{2}a^{\alpha\gamma}\frac{\partial a_{\beta\gamma}}{\partial x^0}; \quad . \quad . \quad . \quad (16)$$

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any Christoffel symbol with two or more zero indices vanishes. All the surviving Christoffel symbols are regular functions of x^0, x^1, x^2, and we may suppose them expanded in power series in x^0. Thus, if we write

$$\left(\frac{\partial^n \mathrm{F}^i_{jk}}{(\partial x^0)^n}\right)_{x^0=0} = \mathrm{F}^i_{jk\,(n)}, \quad \ldots \ldots \quad (17)$$

we have

$$\mathrm{F}^i_{jk} = \sum_{n=0}^{\infty} \frac{1}{n\,!}\,\varepsilon^n \xi^n\,\mathrm{F}^i_{jk\,(n)}, \quad \ldots \ldots \quad (18)$$

where $\mathrm{F}^i_{jk\,(n)}$ are regular functions of x^1, x^2. Functions of derivatives of the Christoffel symbols may be similarly expanded.

Noting that

$$\frac{\partial}{\partial x^0} = \frac{1}{\varepsilon}\frac{\partial}{\partial \xi}, \quad \ldots \ldots \quad (19)$$

we have by (3)

$$\left.\begin{aligned} \mathrm{A}_0 u^0 &= \frac{1}{\varepsilon}\frac{\partial u^0}{\partial \xi}, & \mathrm{A}_0 u^\alpha &= \frac{1}{\varepsilon}\frac{\partial u^\alpha}{\partial \xi} + \mathrm{F}^\alpha_{\beta 0} u^\beta, \\ \mathrm{A}_\alpha u^0 &= \frac{\partial u^0}{\partial x^\alpha} + \mathrm{F}^0_{\alpha\beta} u^\beta, & \mathrm{A}_\alpha u^\beta &= \frac{\partial u^\beta}{\partial x^\alpha} + \mathrm{F}^\beta_{\alpha\gamma} u^\gamma + \mathrm{F}^\beta_{\alpha 0} u^0. \end{aligned}\right\} \quad \ldots \ldots \quad (20)$$

Thus the equation of incompressibility (6) becomes

$$\frac{1}{\varepsilon}\frac{\partial u^0}{\partial \xi} + \frac{\partial u^\alpha}{\partial x^\alpha} + \mathrm{F}^\alpha_{\alpha\gamma} u^\gamma + \mathrm{F}^\alpha_{\alpha 0} u^0 = 0. \quad \ldots \ldots \quad (21)$$

We have also by (4) and (20)

$$\left.\begin{aligned} \mathrm{A}^0\mathrm{A}_0 u^0 &= \frac{1}{\varepsilon^2}\frac{\partial^2 u^0}{\partial \xi^2}, \\ \mathrm{A}^0\mathrm{A}_0 u^\alpha &= \frac{1}{\varepsilon^2}\frac{\partial^2 u^\alpha}{\partial \xi^2} + \frac{2}{\varepsilon}\mathrm{F}^\alpha_{\beta 0}\frac{\partial u^\beta}{\partial \xi} + \left(\frac{\partial}{\partial x^0}\mathrm{F}^\alpha_{\beta 0}\right) u^\beta + \mathrm{F}^\alpha_{\beta 0}\mathrm{F}^\beta_{\gamma 0} u^\gamma, \\ \mathrm{A}^\alpha\mathrm{A}_\alpha u^0 &= a^{\alpha\beta}\Big[-\frac{1}{\varepsilon}\mathrm{F}^0_{\alpha\beta}\frac{\partial u^0}{\partial \xi} + \frac{\partial^2 u^0}{\partial x^\alpha\,\partial x^\beta} - \mathrm{F}^\gamma_{\alpha\beta}\frac{\partial u^0}{\partial x^\gamma} + \mathrm{F}^0_{\beta\gamma}\mathrm{F}^\gamma_{\alpha 0} u^0 \\ &\qquad + 2\mathrm{F}^0_{\alpha\gamma}\frac{\partial u^\gamma}{\partial x^\beta} + \left(\frac{\partial}{\partial x^\beta}\mathrm{F}^0_{\alpha\delta} + \mathrm{F}^0_{\beta\gamma}\mathrm{F}^\gamma_{\alpha\delta} - \mathrm{F}^\gamma_{\alpha\beta}\mathrm{F}^0_{\gamma\delta}\right) u^\delta\Big], \\ \mathrm{A}^\alpha\mathrm{A}_\alpha u^\gamma &= a^{\alpha\beta}\Big[-\frac{1}{\varepsilon}\mathrm{F}^0_{\alpha\beta}\frac{\partial u^\gamma}{\partial \xi} + 2\mathrm{F}^\gamma_{\alpha 0}\frac{\partial u^0}{\partial x^\beta} + \left(\frac{\partial}{\partial x^\beta}\mathrm{F}^\gamma_{\alpha 0} + \mathrm{F}^\gamma_{\beta\delta}\mathrm{F}^\delta_{\alpha 0} - \mathrm{F}^\delta_{\alpha\beta}\mathrm{F}^\gamma_{\delta 0}\right) u^0 \\ &\qquad + \frac{\partial^2 u^\gamma}{\partial x^\alpha\,\partial x^\beta} + 2\mathrm{F}^\gamma_{\alpha\delta}\frac{\partial u^\delta}{\partial x^\beta} - \mathrm{F}^\delta_{\alpha\beta}\frac{\partial u^\gamma}{\partial x^\delta} \\ &\qquad + \left(\frac{\partial}{\partial x^\beta}\mathrm{F}^\gamma_{\alpha\delta} + \mathrm{F}^\gamma_{\beta i}\mathrm{F}^i_{\alpha\delta} - \mathrm{F}^i_{\alpha\beta}\mathrm{F}^\gamma_{i\delta}\right) u^\delta\Big]. \end{aligned}\right\} \quad (22)$$

Thus the equations (9) become

$$\frac{1}{\varepsilon}\frac{\partial p}{\partial \xi} = \frac{\mu}{\varepsilon^2}\frac{\partial^2 u^0}{\partial \xi^2} + \mu a^{\alpha\beta}\left[-\frac{1}{\varepsilon}F^0_{\alpha\beta}\frac{\partial u^0}{\partial \xi} + \frac{\partial^2 u^0}{\partial x^\alpha \partial x^\beta} - F^\gamma_{\alpha\beta}\frac{\partial u^0}{\partial x^\gamma} + F^0_{\beta\gamma}F^\gamma_{\alpha 0}u^0 + 2F^0_{\alpha\gamma}\frac{\partial u^\gamma}{\partial x^\beta} + \left(\frac{\partial}{\partial x^\beta}F^0_{\alpha\gamma} + F^0_{\beta\delta}F^\delta_{\alpha\gamma} - F^\delta_{\alpha\beta}F^0_{\delta\gamma}\right)u^\gamma\right], \quad (23)$$

and

$$a^{\gamma\beta}\frac{\partial p}{\partial x^\beta} = \frac{\mu}{\varepsilon^2}\frac{\partial^2 u^\gamma}{\partial \xi^2} + \frac{2\mu}{\varepsilon}F^\gamma_{\beta 0}\frac{\partial u^\beta}{\partial \xi} - \frac{\mu a^{\alpha\beta}}{\varepsilon}F^0_{\alpha\beta}\frac{\partial u^\gamma}{\partial \xi} + \mu\left(\frac{\partial}{\partial x^0}F^\gamma_{\delta 0} + F^\gamma_{\beta 0}F^\beta_{\delta 0}\right)u^\delta$$
$$+ \mu a^{\alpha\beta}\left[2F^\gamma_{\alpha 0}\frac{\partial u^0}{\partial x^\beta} + \left(\frac{\partial}{\partial x^\beta}F^\gamma_{\alpha 0} + F^\gamma_{\beta\delta}F^\delta_{\alpha 0} - F^\delta_{\alpha\beta}F^\gamma_{\delta 0}\right)u^0 + \frac{\partial^2 u^\gamma}{\partial x^\alpha \partial x^\beta} + 2F^\gamma_{\alpha\delta}\frac{\partial u^\delta}{\partial x^\beta} - F^\delta_{\alpha\beta}\frac{\partial u^\gamma}{\partial x^\delta} + \left(\frac{\partial}{\partial x^\beta}F^\gamma_{\alpha\delta} + F^\gamma_{\beta i}F^i_{\alpha\delta} - F^i_{\alpha\beta}F^\gamma_{i\delta}\right)u^\delta\right]. \quad (24)$$

Equations (21), (23) and (24) are the exact transformation of (6) and (9), the independent variables being now ξ, x^1, x^2. The co-efficients of the type $a^{\alpha\beta}$, and the F's, are to be considered as power series in $\varepsilon\xi$ of the type (18). If these expansions are written in the equations, the equations become explicit in the parameter, but the range of values of ξ, x^1, x^2, for which the equations are valid does not depend on ε.

§ 6 *The formal process of integration. Equations for the principal parts of the solutions.*

We now follow the method of § 3. We wish to find the principal parts of u^i, p, satisfying the equations (21), (23), (24) for $0 \leq \xi \leq \tau(x^1, x^2)$ and for the range of values of x^1, x^2 corresponding to the portion of the surface S covered by the membrane; the boundary conditions are that u^i shall be zero for $\xi = 0$, that for $\xi = \tau$ the values of u^i shall be equal to the components of displacement due to the rigid body displacement of the tooth, and that, in the case of a membrane with an edge, as shown in fig. 1*b*, p shall have an assigned constant value at the edge.

We shall assume that the equations admit (for a range of values of ε) a set of solutions u^i, p, which are expressible in power series in ε, the co-efficients being regular functions of ξ, x^1, x^2. Let us write

$$\left.\begin{aligned} u^0 &= u^0_{(0)} + \varepsilon u^0_{(1)} + \varepsilon^2 u^0_{(2)} + \dots, \\ u^\alpha &= u^\alpha_{(0)} + \varepsilon u^\alpha_{(1)} + \varepsilon^2 u^\alpha_{(2)} + \dots, \\ p &= p_{(0)} + \varepsilon p_{(1)} + \varepsilon^2 p_{(2)} + \dots. \end{aligned}\right\} \quad (25)$$

Let us write the boundary conditions as follows:—

$$\left.\begin{aligned} &\text{for } \xi = 0, \quad u^i = 0: \\ &\text{for } \xi = \tau, \quad u^i = \varepsilon\beta^i_{(1)} + \varepsilon^2\beta^i_{(2)} + \varepsilon^3\beta^i_{(3)} + \dots: \\ &\text{for } f(x^1, x^2) = 0, \quad p = P, \end{aligned}\right\} \quad (26)$$

where $f(x^1, x^2) = 0$ is the margin of the membrane (if it has one). We have omitted, for $\xi = \tau$, terms independent of ε, since the displacement of the tooth is certainly to tend to zero with ε. $\beta^i_{(n)}$ are vectors whose components are functions of x^1, x^2 only.

The formal method of solution is the following. We substitute from (25) in (21), (23), (24), and thus obtain four equations of the type

$$X_{(-2)}\,\varepsilon^{-2} + X_{(-1)}\varepsilon^{-2} + X_{(0)} + X_{(1)}\,\varepsilon + X_{(2)}\,\varepsilon^2 + \ldots = 0, \quad \ldots\ldots \quad (27)$$

in which the X's are expressions in u^i, p and their derivatives with respect to ξ, x^1, x^2. Since (27) is to hold for a range of values of ε, each X must vanish separately. Hence we get a sequence of equations for $u^i_{(n)}$, $p_{(n)}$, with the boundary conditions

$$\left.\begin{array}{l} \text{for } \xi = 0, \quad u^i_{(n)} = 0, \quad (n = 0, 1, 2, \ldots): \\ \text{for } \xi = \tau, \quad u^i_{(0)} = 0, \quad u^i_{(n)} = \beta^i_{(n)}, \ (n = 1, 2, 3, \ldots): \\ \text{for } f(x^1, x^2) = 0, \quad p_{(0)} = \mathrm{P},\ p_{(n)} = 0, \quad (n = 1, 2, 3, \ldots). \end{array}\right\} \quad (28)$$

Equivalently, we are to pick out and equate to zero the coefficients of the various powers of ε in (21), (23), (24). Thus we obtain from (21) the three following equations :—

$$\frac{\partial u^{(0)}_0}{\partial \xi} = 0, \quad \ldots\ldots \quad (29)$$

$$\frac{\partial u^0_{(1)}}{\partial \xi} + \frac{\partial u^\alpha_{(0)}}{\partial x^\alpha} + \mathrm{F}^\alpha_{\alpha\gamma(0)}\, u^\gamma_{(0)} + \mathrm{F}^\alpha_{\alpha 0\,(0)}\, u^0_{(0)} = 0, \quad \ldots\ldots \quad (30)$$

$$\frac{\partial u^0_{(2)}}{\partial \xi} + \frac{\partial u^\alpha_{(1)}}{\partial x^\alpha} + \xi \mathrm{F}^\alpha_{\alpha\gamma(1)}\, u^\gamma_{(0)} + \mathrm{F}^\alpha_{\alpha\gamma(0)}\, u^\gamma_{(1)} + \xi \mathrm{F}^\alpha_{\alpha 0\,(1)}\, u^0_{(0)} + \mathrm{F}^\alpha_{\alpha 0\,(0)}\, u^0_{(1)} = 0. \quad \ldots \quad (31)$$

It will be remembered that the F's occurring here are functions of x^1 x,^{2} only. Also from (23)

$$\frac{\partial p_{(0)}}{\partial \xi} = \mu \frac{\partial^2 u^0_{(1)}}{\partial \xi^2} - \mu a^{\alpha\beta}_{(0)} \mathrm{F}^0_{\alpha\beta(0)} \frac{\partial u^0_{(0)}}{\partial \xi}, \quad \ldots\ldots \quad (32)$$

where we adopt the notation $a^{\alpha\beta}_{(0)}$ to indicate that the value is calculated for $\xi = 0$. Also from (24)

$$\frac{\partial^2 u^\gamma_{(0)}}{\partial \xi^2} = 0. \quad \ldots\ldots \quad (33)$$

When we combine (29) with the boundary conditions (28) we get

$$u^0_{(0)} = 0, \quad \ldots\ldots \quad (34)$$

while (33) and (28) give

$$u^\gamma_{(0)} = 0. \quad \ldots\ldots \quad (35)$$

We now deduce from (30) that $u^0_{(1)}$ is a function of x^1, x^2 only. This is inconsistent with (28) unless $\beta^0_{(1)} = 0$. *Therefore, in order that solutions of the type considered may exist, it is necessary that* $\beta^0_{(1)} = 0$. We shall assume this to be the case. Now $\beta^i_{(1)}$ is a vector, defined by the set of rigid body displacements of the tooth T corresponding to the values of ε in the range; its component $\beta^0_{(1)}$, normal to S, vanishes over S. *Hence, in general, its other components must also vanish, and we must put*

$$\beta^i_{(1)} = 0. \quad \ldots\ldots \quad (36)$$

Then

$$u^0_{(1)} = 0. \quad \ldots \ldots \ldots \ldots \ldots \quad (37)$$

We deduce from (32)

$$\frac{\partial p_{(0)}}{\partial \xi} = 0, \; p_{(0)} = p_{(0)}(x^1, x^2). \quad \ldots \ldots \ldots \quad (38)$$

Also (24) gives, by virtue of (35),

$$\frac{\partial^2 u^\gamma_{(1)}}{\partial \xi^2} = 0, \quad \ldots \ldots \ldots \ldots \ldots \quad (39)$$

so that, by (28) and (36),

$$u^\gamma_{(1)} = 0. \quad \ldots \ldots \ldots \ldots \ldots \quad (40)$$

But (31) now gives

$$\frac{\partial u^0_{(2)}}{\partial \xi} = 0, \quad \ldots \ldots \ldots \ldots \ldots \quad (41)$$

so that, in view of (28), we must take $\beta^0_{(2)} = 0$, and hence, by the same argument as that prior to (36),

$$\beta^i_{(2)} = 0. \quad \ldots \ldots \ldots \ldots \ldots \quad (42)$$

We have, in consequence,

$$u^0_{(2)} = 0. \quad \ldots \ldots \ldots \ldots \ldots \quad (43)$$

Collecting our results, we have

$$u^i_{(0)} = 0, \quad u^i_{(1)} = 0, \quad u^0_{(2)} = 0, \quad p_{(0)} = p_{(0)}(x^1, x^2), \quad \beta^i_{(1)} = 0, \; \beta^i_{(2)} = 0, \quad (44)$$

and by (24)

$$a^{\gamma\beta}_{(0)} \frac{\partial p_{(0)}}{\partial x^\beta} = \mu \frac{\partial^2 u^\gamma_{(2)}}{\partial \xi^2}, \quad \ldots \ldots \ldots \ldots \quad (45)$$

while equating to zero the coefficient of ε^2 in (21) we have

$$\frac{\partial u^0_{(3)}}{\partial \xi} + \frac{\partial u^\alpha_{(2)}}{\partial x^\alpha} + F^\alpha_{\alpha\gamma(0)} u^\gamma_{(2)} = 0. \quad \ldots \ldots \ldots \ldots \quad (46)$$

By (45) we see that if we are to get a finite change in pressure as we pass over the surface S in the limit when ε tends to zero, $u^\gamma_{(2)}$ must not vanish identically. Hence we may state the following result :—

Theorem I.—*If a tooth T, separated from a socket S by a thin membrane of finite rigidity, whose thickness is of the order of* ε, *is subjected to finite forces, the tooth undergoes a displacement which is of the order of* ε^3; *inside the membrane the tangential displacement is of the order of* ε^2, *while the normal displacement is of the order of* ε^3.

The principal part of the pressure ($p_{(0)}$), the principal part of the tangential displacement ($\varepsilon^2 u^\gamma_{(2)}$) and the principal part of the normal displacement ($\varepsilon^3 u^0_{(3)}$) are such that the equations (45) and (46) are satisfied, the boundary conditions being

$$\left.\begin{array}{lll} \text{for } \xi = 0, & u^\gamma_{(2)} = 0, & u^0_{(3)} = 0 : \\ \text{for } \xi = \tau, & u^\gamma_{(2)} = 0, & u^0_{(3)} = \beta^0_{(3)} : \\ \text{for } f(x^1, x^2) = 0, & p_{(0)} = P : & \end{array}\right\} \quad \ldots \ldots \ldots \quad (47)$$

$\varepsilon^3 \beta^0_{(3)}$ is the principal part of the normal component of the displacement of a point of the surface of the tooth T, considered as a function of x^1, x^2, and measured in the sense from the socket S to T.

§ 7. *The principal parts of the displacement, and the partial differential equation for the pressure.*

Integrating (45) and using (47), we obtain

$$u^{\gamma}_{(2)} = \frac{1}{2\mu}\,\xi\,(\xi - \tau)\,a^{\gamma\beta}_{(0)}\,\frac{\partial p_{(0)}}{\partial x^{\beta}}, \quad \ldots \ldots \quad (48)$$

since $p_{(0)}$ is a function of x^1, x^2 only. (46) then gives, with the boundary condition at $\xi = 0$ from (47),

$$u^0_{(3)} = -\frac{1}{2\mu}\left(\tfrac{1}{3}\xi^3 - \tfrac{1}{2}\xi^2\tau\right)\left[\frac{\partial}{\partial x^{\gamma}}\left(a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right) + F^{\alpha}_{\alpha\gamma(0)}a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right] + \frac{1}{4\mu}\,\xi^2 a^{\gamma\beta}_{(0)}\frac{\partial \tau}{\partial x^{\gamma}}\frac{\partial p_{(0)}}{\partial x^{\beta}}\,; \quad (49)$$

there remains the boundary condition at $\xi = \tau$ to be satisfied, and this gives

$$\beta^0_{(3)} = \frac{\tau^3}{12\mu}\left[\frac{\partial}{\partial x^{\gamma}}\left(a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right) + F^{\alpha}_{\alpha\gamma(0)}a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right] + \frac{\tau^2}{4\mu}\,a^{\gamma\beta}_{(0)}\frac{\partial \tau}{\partial x^{\gamma}}\frac{\partial p_{(0)}}{\partial x^{\beta}}. \quad \ldots \quad (50)$$

Now let the line-element of S be

$$d\sigma^2 = b_{\alpha\beta}dx^{\alpha}dx^{\beta} = a_{\alpha\beta(0)}dx^{\alpha}dx^{\beta}\,; \quad \ldots \ldots \quad (51)$$

we have then

$$b_{\alpha\beta} = a_{\alpha\beta(0)}, \quad b^{\alpha\beta} = a^{\alpha\beta}_{(0)}, \quad \ldots \ldots \quad (52)$$

and if $G^{\alpha}_{\beta\gamma}$ are the Christoffel symbols corresponding to $b_{\alpha\beta}$, we have

$$G^{\alpha}_{\beta\gamma} = F^{\alpha}_{\beta\gamma(0)}. \quad \ldots \ldots \quad (53)$$

If, then, we denote by B_{α} the operation of covariant differentiation with respect to $b_{\alpha\beta}$, and write

$$B^{\alpha} = b^{\alpha\beta}\,B_{\beta},$$

we have

$$\begin{aligned}\frac{\partial}{\partial x^{\gamma}}\left(a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right) + F^{\alpha}_{\alpha\gamma(0)}a^{\gamma\beta}_{(0)}\frac{\partial p_{(0)}}{\partial x^{\beta}} &= \frac{\partial}{\partial x^{\gamma}}\left(b^{\gamma\beta}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right) + G^{\alpha}_{\alpha\gamma}b^{\gamma\beta}\frac{\partial p_{(0)}}{\partial x^{\beta}}\\ &= B_{\gamma}\left(b^{\gamma\beta}\frac{\partial p_{(0)}}{\partial x^{\beta}}\right)\\ &= B_{\gamma}B^{\gamma}p_{(0)} \quad \ldots \ldots \quad (54)\end{aligned}$$

Thus (50) may be written

$$\beta^0_{(3)} = \frac{\tau^3}{12\mu}B^{\alpha}B_{\alpha}p_{(0)} + \frac{\tau^2}{4\mu}(B^{\alpha}\tau)(B_{\alpha}p_{(0)}) = \frac{1}{12\mu}B^{\alpha}(\tau^3 B_{\alpha}p_{(0)}). \quad \ldots \quad (55)$$

Introducing infinitesimals again, since there is now no danger of confusion, we may state the following result :—

THEOREM II.—*The principal parts of the tangential components of the displacement in the membrane are given by*

$$u^\gamma = (1/2\mu)\, x^0\, (x^0 - h)\, B^\gamma p, \qquad (56)$$

and the principal part of the normal component is given by

$$u^0 = -(1/2\mu)\,[\tfrac{1}{3}(x^0)^3 - \tfrac{1}{2}h\,(x^0)^2]\, B^\alpha B_\alpha p + (1/4\mu)\,(x^0)^2\,(B^\alpha h)\,(B_\alpha p), \qquad (57)$$

where μ *is the rigidity of the membrane,* h *is its infinitesimal thickness* ($h = \varepsilon\tau$), x^0 *is the normal distance of a point in the membrane from the socket* S, B_α *and* B^α *are covariant and contravariant operators with respect to the metric of* S, *and* p *is the principal part of the pressure in the membrane ;* p *satisfies the partial differential equation*

$$B^\alpha\,(h^3 B_\alpha p) = 12\mu\beta, \qquad (58)$$

where β *is the principal part of the normal component of the displacement of the surface of the tooth* T, *in the sense from* S *to* T, *considered as a function of arbitrarily selected curvilinear co-ordinates on the surface of the socket* S, *or on the surface of the tooth* T.*

Probably the most useful form for the equation (58) is

$$\frac{1}{\sqrt{b}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{b}\,.\,h^3 b^{\alpha\beta}\frac{\partial p}{\partial x^\beta}\right) = 12\mu\beta, \qquad (59)$$

where b denotes the determinant $|b_{\alpha\beta}|$.

As has been mentioned, there are two types of problem, namely, that of a closed membrane, fig. 1*a*, and that of an open membrane, fig. 1*b*, the margin being subject to a specified pressure P. In the former, the pressure remains undetermined to an additive constant ; in the latter that constant is determined. In either case we may superimpose results ; that is to say, we may express β as the sum of normal displacements $\beta^{(1)}, \ldots \beta^{(6)}$, caused by the several constituents of the rigid body displacement of the tooth, and find p for each displacement separately, adding together the results, and, if necessary, choosing the additive constant suitably.

If the surface S is *developable*, it is possible to find a system of surface co-ordinates x^1, x^2 such that the line-element of S is

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2.$$

For these co-ordinates (59) takes the form

$$\frac{\partial}{\partial x^1}\left(h^3\frac{\partial p}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(h^3\frac{\partial p}{\partial x^2}\right) = 12\mu\beta. \qquad (60)$$

* The above analytical results agree (except of course for the tensor notation) with those first given by REYNOLDS (*loc. cit.*). His argument was intuitive, and it is not at all clear, without some such argument as that just given, that the curvature of surfaces does not enter in some way other than in the left-hand side of (58), where it is implicit.

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If the surface S is a *surface of revolution*, we may conveniently take x^1 to be the length of a meridian curve, measured to a general point from some specified section normal to the axis of revolution, and x^2 to be the azimuthal angle. If, then, R is the radius of the circular section, normal to the axis of revolution and passing through a general point of the surface S, we have

$$d\sigma^2 = (dx^1)^2 + \mathrm{R}^2 (dx^2)^2, \quad \ldots \ldots \ldots \ldots \quad (61)$$

where R is a function of x^1 only. Equation (59) becomes

$$\frac{1}{\mathrm{R}} \frac{\partial}{\partial x^1}\left(\mathrm{R}h^3 \frac{\partial p}{\partial x^1}\right) + \frac{1}{\mathrm{R}^2} \frac{\partial}{\partial x^2}\left(h^3 \frac{\partial p}{\partial x^2}\right) = 12\mu\beta. \quad \ldots \ldots \ldots \quad (62)$$

Part II.—The Two-Dimensional Problem.

§ 8. *Statement of the problem. Determination of the pressure in the membrane in terms of the displacement of the tooth.*

The problem of the tooth is, of course, actually a three-dimensional problem, but we shall here investigate the analogous two-dimensional problem, which may be stated as follows :—

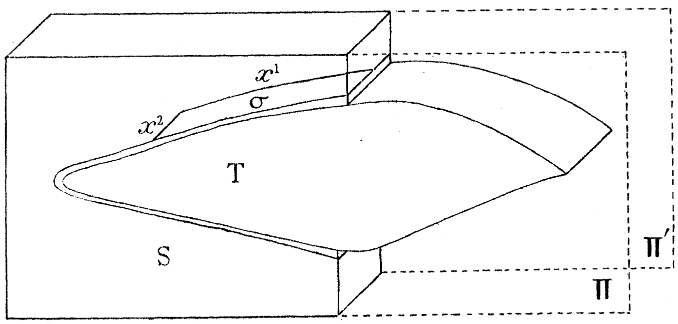


Fig. 3.

Let the socket S and the tooth T be cylinders with parallel generators; let these two cylinders and the membrane between them be terminated by two fixed planes Π, Π′, perpendicular to the generators. As usual, S is fixed; T can move with a two-dimensional motion parallel to the fixed planes, which we shall suppose to form smooth constraints for T. The boundary conditions will be as follows :—

(i) zero displacement on S;

(ii) assigned rigid body displacement on T;

(iii) assigned pressure P on the margin of the membrane, which we shall suppose to correspond to generators of S;

(iv) zero displacement at the planes Π, Π′ in the direction of the generators S.

Since the displacement considered in (iv) is what we have called a tangential displacement in the membrane, which (*cf.* Theorem I) is generally of the second order, it is a matter of indifference whether we leave (iv) as above, or state that the displacement in question shall be of the third order.

Let us introduce a co-ordinate system x^1, x^2 on S, such that x^1 is the distance to a general point of S from an assigned generator, measured along the intersection of S by a plane parallel to Π, and x^2 is the perpendicular distance from the plane Π.

For the determination of the pressure p in the membrane we have (60), which, since h is a function of x^1 only, takes the form

$$\frac{\partial}{\partial x^1}\left(h^3 \frac{\partial p}{\partial x^1}\right) + h^3 \frac{\partial^2 p}{(\partial x^2)^2} = 12\mu\beta. \quad (63)$$

The boundary conditions on Π, Π' are, by (56),

$$\partial p / \partial x^2 = 0. \quad (64)$$

It is evident that p will be a function of x^1 only; if we write $\sigma = x^1$, so that σ denotes the arc of the section of S by Π, we see that *in the general two-dimensional problem, the pressure p satisfies the differential equation*

$$\frac{d}{d\sigma}\left(h^3 \frac{dp}{d\sigma}\right) = 12\mu\beta, \quad (65)$$

with the boundary condition $p = P$ on the margin of the membrane, which we may suppose to correspond to $\sigma = 0$ and $\sigma = l$.

The thickness h of the membrane is, of course, supposed to be a known function of σ.

It is clear that the problem is a purely two-dimensional one, there being, by (56), no displacement parallel to the generators of S. We may therefore employ the language of two dimensions, and refer to S and T as curves instead of surfaces.

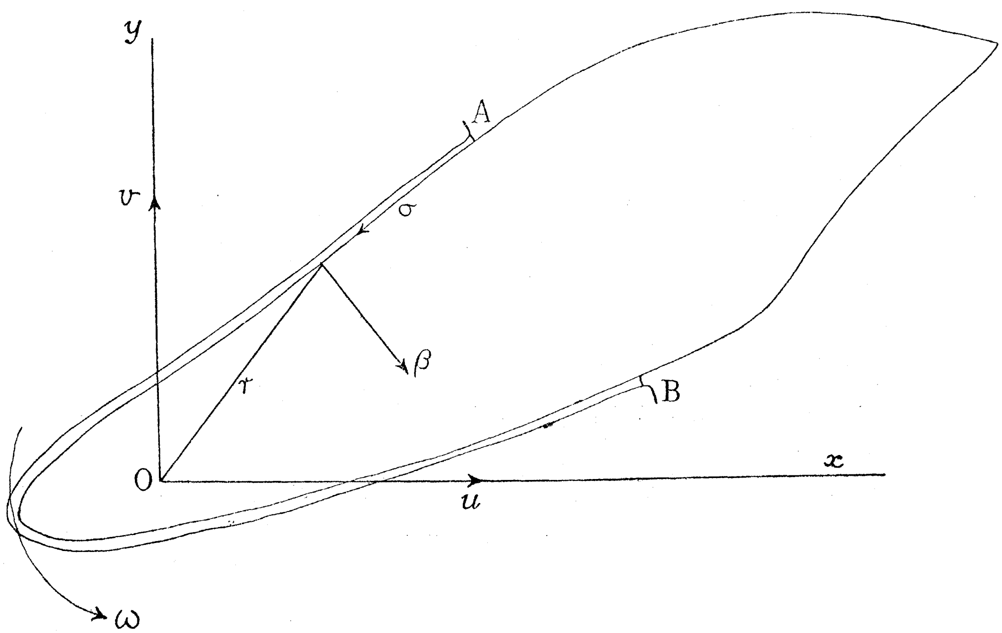


FIG. 4.

Let us now take rectangular axes Oxy in the plane of the displacement. Let the displacement of the tooth T be reduced to an infinitesimal translation with components u, v along the axes of co-ordinates, and an infinitesimal rotation ω about O. The components of displacement of a point M of T are then

$$u - y\omega, \qquad v + x\omega,$$

and for the normal component β, in the sense from S to T, we have

$$\beta = -u\frac{dy}{d\sigma} + v\frac{dx}{d\sigma} + \omega r\frac{dr}{d\sigma}, \qquad (r^2 = x^2 + y^2), \quad \ldots \ldots \quad (66)$$

where these derivatives are calculated for the curve S, whose equations are supposed to be known in the form $x = x(\sigma)$, $y = y(\sigma)$; we suppose the axes Oxy to be right-handed (*i.e.*, the rotation from Ox to Oy is counter-clockwise) and the direction in which σ is measured is such that as we travel along S in the sense of σ increasing, T lies on our left-hand side.

Substituting from (66) in (65) and integrating, we have this result:

THEOREM III.—*In the general two-dimensional problem, the pressure gradient along the membrane is given by*

$$h^3\frac{dp}{d\sigma} = 12\mu\left(-uy + vx + \tfrac{1}{2}\omega r^2\right) - C, \quad \ldots \ldots \ldots \quad (67)$$

where h is the thickness of the membrane, u, v, ω the components of the displacement of the tooth, $r^2 = x^2 + y^2$, and C is a constant of integration.

Let us denote by A and B, fig. 4, the margin of the membrane, corresponding respectively to $\sigma = 0$ and $\sigma = l$, and let M be a general point. Then since $p = P$ at A, integration of (67) gives

$$p - P = -12\mu u\int_A^M yh^{-3}\,d\sigma + 12\mu v\int_A^M xh^{-3}\,d\sigma + 6\mu\omega\int_A^M r^2h^{-3}\,d\sigma - C\int_A^M h^{-3}\,d\sigma. \quad (68)$$

The value of the constant C is given by the boundary condition $p = P$ at B; we have in fact

$$C\int_A^B h^{-3}\,d\sigma = -12\mu u\int_A^B yh^{-3}\,d\sigma + 12\mu v\int_A^B xh^{-3}\,d\sigma + 6\mu\omega\int_A^B r^2h^{-3}\,d\sigma. \qquad (69)$$

If the membrane forms a closed cylinder, so that there is no margin, instead of two boundary conditions we have only one, corresponding to the fact that p is single-valued. The pressure p is again given by (68), but now P is an undetermined constant, while C is given by (69), in which $\int_A^B$ is to be understood as an integration taken right round the closed curve S.

We shall find it convenient to introduce the following notation :

$$\left.\begin{aligned}
&K^{(0)} = \int_A^B h^{-3}\,d\sigma, \qquad K_x^{(1)} = \int_A^B xh^{-3}\,d\sigma, \qquad K_y^{(1)} = \int_A^B yh^{-3}\,d\sigma,\\
&K_{xx}^{(2)} = \int_A^B x^2h^{-3}\,d\sigma, \qquad K_{xy}^{(2)} = \int_A^B xyh^{-3}\,d\sigma, \qquad K_{yy}^{(2)} = \int_A^B y^2h^{-3}\,d\sigma,\\
&K^{(2)} = K_{xx}^{(2)} + K_{yy}^{(2)} = \int_A^B r^2h^{-3}\,d\sigma,\\
&K_x^{(3)} = \int_A^B xr^2h^{-3}\,d\sigma, \qquad K_y^{(3)} = \int_A^B yr^2h^{-3}\,d\sigma, \qquad K^{(4)} = \int_A^B r^4h^{-3}\,d\sigma.
\end{aligned}\right\} \tag{70}$$

Rewriting (68) we have the result :

THEOREM IV.—*In the general two-dimensional problem, the pressure p at any point* M *of the membrane is given by*

$$\begin{aligned}
(p - P)\,K^{(0)} = &- 12\mu u\left(K^{(0)}\int_A^M yh^{-3}\,d\sigma - K_y^{(1)}\int_A^M h^{-3}\,d\sigma\right)\\
&+ 12\mu v\left(K^{(0)}\int_A^M xh^{-3}\,d\sigma - K_x^{(1)}\int_A^M h^{-3}\,d\sigma\right)\\
&+ 6\mu\omega\left(K^{(0)}\int_A^M r^2h^{-3}\,d\sigma - K^{(2)}\int_A^M h^{-3}\,d\sigma\right), \quad \ldots \ldots \ldots
\end{aligned} \tag{71}$$

where P is the atmospheric pressure and u, v , ω are the components of the displacement of the tooth.

§ 9. *Points of maximum and minimum pressure.*

So far, both the origin and directions of the rectangular axes have been arbitrary. *Let us now temporarily choose for origin* O *the centre of rotation of the infinitesimal rigid body displacement of the tooth.* (This can always be done unless the displacement of the tooth is a pure translation, a case which we shall discuss later.) We have then

$$u = v = 0, \quad \ldots \ldots \ldots \ldots \tag{72}$$

and therefore (71) reads

$$p - P = 6\mu\omega\left(\int_A^M r^2h^{-3}\,d\sigma - \frac{K^{(2)}}{K^{(0)}}\int_A^M h^{-3}\,d\sigma\right), \quad \ldots \ldots \ldots \tag{73}$$

and the pressure gradient is

$$\frac{dp}{d\sigma} = 6\mu\omega h^{-3}\left(r^2 - \frac{K^{(2)}}{K^{(0)}}\right). \quad \ldots \ldots \ldots \tag{74}$$

Referring to (70), we see that $K^{(2)}/K^{(0)}$ has a simple interpretation ; if we replace the section of the membrane by a fictitious wire whose linear density is equal to $1/h^3$, then $K^{(2)}/K^{(0)}$ is equal to the square of the radius of gyration of this fictitious wire about the centre of rotation O. Accordingly, by (74) and (65), we may state this result :

THEOREM V.—*In the general two-dimensional problem, the points of the membrane at which the pressure p has maximum and minimum values are situated on a circle whose* *centre is the centre of rotation of the tooth, and whose radius is equal to the radius of gyration about that centre of rotation of a fictitious wire coincident with the section of the membrane by the plane of the displacement, and having a linear density equal to the cube of the reciprocal of the thickness of the membrane.*

The points of maximum pressure are distinguished from the points of minimum pressure by the fact that at a point of maximum pressure, the tooth approaches the socket as a result of the displacement, while at a point of minimum pressure, the tooth recedes from the socket.

These results are illustrated in fig. 5, in which the arrows indicate the directions in which the pressure increases.

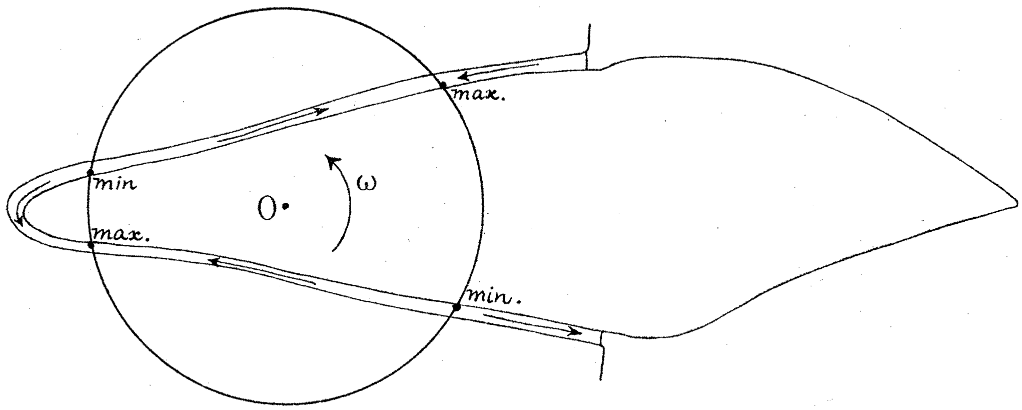


Fig. 5.

If the displacement of the tooth is a *pure translation*, we cannot choose the origin at the centre of rotation, which is at infinity. *It is convenient to choose the origin temporarily at the centre of mass of the fictitious wire*, so that by (70)

$$K_x^{(1)} = K_y^{(1)} = 0. \qquad (75)$$

Since ω is zero in the present case, (71) becomes

$$p - P = 12\mu \int_A^M (xv - yu)\, h^{-3} d\sigma, \qquad (76)$$

and the pressure gradient is

$$dp/d\sigma = 12\mu h^{-3} (xv - yu). \qquad (77)$$

Hence we may state the result :

Theorem VI.—*If, in the two-dimensional problem, the tooth receives a pure translation, the points of maximum and minimum pressure occur at the points where the membrane is cut by a straight line, drawn parallel to the direction of the translation through the centre of mass of the fictitious wire previously described. The pressure is a maximum or a minimum according as the tooth approaches or recedes from the socket.*

This result is illustrated in fig. 6, in which the arrows indicate the directions in which the pressure increases.

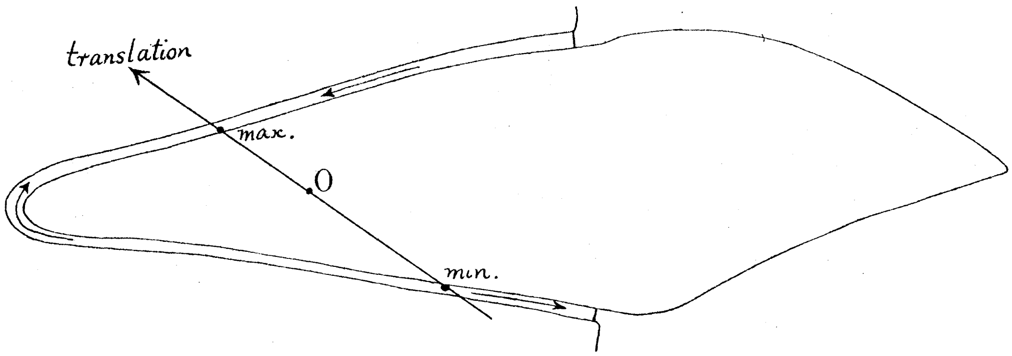


FIG. 6.

§ 10. *Connection between the applied force-system and the displacement of the tooth; centre of rotation.*

Let us now consider the applied force-system which produces the displacement of the tooth. We shall suppose that the part of the tooth outside the membrane is subject to the uniform atmospheric pressure P; this pressure will not be counted as an applied force.

Taking again general rectangular axes in the plane of the displacement, the applied force-system may be reduced to components X, Y along the axes and a couple N. Since the principal part of the elastic traction on the tooth arises from the pressure p alone, and since a uniform pressure P has zero resultant, the conditions for the statical equilibrium of the tooth are easily seen to be

$$X = b\int_A^B (p - P)\,dy,\ Y = -b\int_A^B (p - P)\,dx,\ N = -b\int_A^B (p - P)\,(x\,dx + y\,dy), \quad (78)$$

where b is the distance between the parallel planes Π, Π' bounding the membrane, or, in other words, the width of the tooth. Using the boundary conditions $p = P$ at A and B, we obtain by integration by parts

$$X = -b\int_A^B y\,dp, \qquad Y = b\int_A^B x\,dp, \qquad N = \tfrac{1}{2}b\int_A^B r^2\,dp. \quad . \quad . \quad . \quad . \quad (79)$$

Substituting for p from (71), we obtain the following result:

THEOREM VII.—*If, in the general two-dimensional problem, the applied force-system* X, Y, N, *produces a displacement* u, v, ω, *in the tooth, then the force-system is connected with the displacement by the equations*

$$\left.\begin{aligned}
XK^{(0)}/b &= 12\mu u\,[K^{(0)}K^{(2)}_{yy} - (K^{(1)}_y)^2] - 12\mu v\,[K^{(0)}K^{(2)}_{xy} - K^{(1)}_x K^{(1)}_y] \\
&\qquad - 6\mu\omega\,[K^{(0)}K^{(3)}_y - K^{(2)}K^{(1)}_y], \\
YK^{(0)}/b &= -12\mu u\,[K^{(0)}K^{(2)}_{xy} - K^{(1)}_x K^{(1)}_y] + 12\mu v\,[K^{(0)}K^{(2)}_{xx} - (K^{(1)}_x)^2], \\
&\qquad + 6\mu\omega\,[K^{(0)}K^{(3)}_x - K^{(2)}K^{(1)}_x], \\
NK^{(0)}/b &= -6\mu u\,[K^{(0)}K^{(3)}_y - K^{(2)}K^{(1)}_y] + 6\mu v\,[K^{(0)}K_x^{(3)} - K^{(2)}K^{(1)}_x] \\
&\qquad + 3\mu\omega\,[K^{(0)}K^{(4)} - (K^{(2)})^2],
\end{aligned}\right\} \quad (80)$$

where the K*'s are geometrical constants of the tooth and membrane, given by* (70).

The above equations may, of course, be solved for the components of displacement u, v, ω, in terms of the force-system X, Y, N, but on account of the complexity of the expressions, we shall not trouble to write them down.

We may simplify (80) by a special choice of axes. If we choose the origin at the centre of rotation, we have $u = v = 0$, and the first two terms on the right-hand side of each of the equations disappear. But this is not a very useful choice of origin, because it depends on the displacement and therefore on the applied force-system. It is better to choose temporarily the origin at the centre of mass of the fictitious wire, and the axes in the directions of the principal axes of inertia of the wire with respect to that point. We have then

$$K_x^{(1)} = K_y^{(1)} = K_{xy}^{(2)} = 0, \qquad (81)$$

and hence, simplifying (80), we may state the result:

THEOREM VIII.—*If, in the general two-dimensional problem, the co-ordinate axes coincide with the principal axes of inertia of the fictitious wire previously described at its centre of mass, then the applied force-system is connected with the displacement of the tooth by the equations*

$$\left.\begin{aligned} X/b &= 12\mu u K_{yy}^{(2)} - 6\mu\omega K_y^{(3)}, \\ Y/b &= 12\mu v K_{xx}^{(2)} + 6\mu\omega K_x^{(3)}, \\ N/b &= -6\mu u K_y^{(3)} + 6\mu v K_x^{(3)} + 3\mu\omega\left[K^{(4)} - \frac{(K^{(2)})^2}{K^{(0)}}\right], \end{aligned}\right\} \qquad (82)$$

where the K'*s are given by* (70).

§ 11. *The wedge-shaped model.*

The simplest geometrical form which we can assign to the two-dimensional root, consistent with a rough approximation to reality, is that of a wedge. We shall accordingly discuss the problem of the wedge-shaped model root, assuming for simplicity that the membrane is of uniform thickness h.

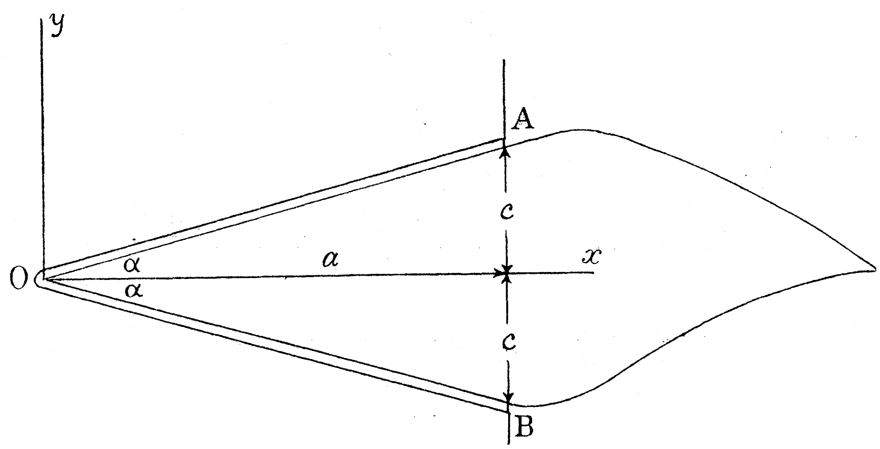


FIG. 7.

We shall choose the origin at the apex of the root, and the axis of x along its central line. We shall adopt the following notation, the numerical values being those which we shall employ in making calculations for the upper central tooth (*cf.* § 1) :

$$\left.\begin{array}{l} \text{Length of root} = a = 0\cdot 49 \text{ in.;} \\ \text{Thickness} = \text{AB} = 2c = 0\cdot 27 \text{ in.;} \\ \text{Width} = b = 0\cdot 24 \text{ in.}^{*}; \\ \text{Angle of root} = 2\alpha; \quad \alpha = \tan^{-1}(c/a) = 15^\circ\, 25'; \\ \text{Distance of biting edge from apex} = \rho = 0\cdot 88 \text{ in.;} \\ \text{Area of rectangular cross-section at margin} = \text{A} = 2bc = 0\cdot 065 \text{ sq. in.}^{\dagger} \end{array}\right\} \quad (83)$$

Calculating the values of the K's from (70), we obtain the following values :

$$\left.\begin{array}{lll} \text{K}^{(0)} = 2ah^{-3}\sec\alpha, & \text{K}^{(1)}_x = a^2h^{-3}\sec\alpha, & \text{K}^{(1)}_y = 0, \\ \text{K}^{(2)}_{xx} = \tfrac{2}{3}a^3h^{-3}\sec\alpha, & \text{K}^{(2)}_{xy} = 0, & \text{K}^{(2)}_{yy} = \tfrac{2}{3}a^3h^{-3}\sec\alpha\tan^2\alpha, \\ \text{K}^{(2)} = \tfrac{2}{3}a^3h^{-3}\sec^3\alpha, & & \\ \text{K}^{(3)}_x = \tfrac{1}{2}a^4h^{-3}\sec^3\alpha, & \text{K}^{(3)}_y = 0, & \text{K}^{(4)} = \tfrac{2}{5}a^5h^{-3}\sec^5\alpha. \end{array}\right\} \quad (84)$$

Substituting these values in (80), we obtain the following result :

THEOREM IX.—*For the wedge-shaped model, the applied force-system is given in terms of the displacement of the tooth by the following formulæ* :

$$\left.\begin{array}{l} \text{X} = 8\mu u a^3bh^{-3}\sec\alpha\tan^2\alpha, \\ \text{Y} = \mu a^3bh^{-3}\sec\alpha\,(2v + a\omega\sec^2\alpha), \\ \text{N} = \mu a^4bh^{-3}\sec^3\alpha\,(v + \tfrac{8}{15}a\omega\sec^2\alpha) : \end{array}\right\} \quad (85)$$

the displacement of the tooth is given in terms of the applied force-system by the formulæ :

$$\left.\begin{array}{l} u = \dfrac{h^3\cos\alpha\cot^2\alpha}{8\mu a^3b}\,\text{X}, \\ v = \dfrac{h^3\cos\alpha}{\mu a^3b}\left(8\text{Y} - 15\,\dfrac{\text{N}}{a}\cos^2\alpha\right), \\ \omega = \dfrac{15h^3\cos^3\alpha}{\mu a^4b}\left(2\,\dfrac{\text{N}}{a}\cos^2\alpha - \text{Y}\right). \end{array}\right\} \quad (86)$$

Hence the co-ordinates of the *centre of rotation* are

$$\left.\begin{array}{l} x = -\dfrac{v}{\omega} = \dfrac{a}{15\cos^2\alpha}\,\dfrac{15\,(\text{N}/a)\cos^2\alpha - 8\text{Y}}{2\,(\text{N}/a)\cos^2\alpha - \text{Y}}, \\ y = \dfrac{u}{\omega} = \dfrac{a}{120\sin^2\alpha}\,\dfrac{\text{X}}{2\,(\text{N}/a)\cos^2\alpha - \text{Y}}. \end{array}\right\} \quad (87)$$

* We shall adopt this numerical value for the purpose of estimating the order of magnitude of results, while recognising the obvious insufficiency of the two-dimensional theory as a representation of reality.

† The area of the actual cross-section, if we suppose it elliptical, is 0·051 sq. in.

If the applied force-system consists of a single force, whose line of action cuts the axis of the root at a distance ρ from the apex, we have

$$N = \rho Y,$$

and the co-ordinates of the centre of rotation become

$$\left.\begin{aligned} x &= \frac{a}{15\cos^2\alpha}\,\frac{15(\rho/a)\cos^2\alpha - 8}{2(\rho/a)\cos^2\alpha - 1}, \\ y &= \frac{a}{120\sin^2\alpha}\,\frac{1}{2(\rho/a)\cos^2\alpha - 1}\,\frac{X}{Y}. \end{aligned}\right\} \qquad (88)$$

We note that the abscissa x of the centre of rotation depends (for a given wedge-shaped root) only on the distance from the apex to the point where the line of action of the applied force cuts the axis of the root, whereas the ordinate y depends, not only on this distance, but also on the inclination of the applied force to the axis.

To get a definite result for the wedge-shaped model of the upper central tooth, let us suppose that the single force is applied at the biting edge of the tooth, which we shall suppose to lie on the axis of the root produced. We are then to put $\rho = 0\cdot88$ in., and therefore $\rho/a = 1\cdot80$. Equations (88) then give, *for the centre of rotation of the wedge-shaped model of the upper central tooth*

$$x = 0\cdot52a = 0\cdot25 \text{ in.}, \quad y = 0\cdot050a\,(X/Y) = 0\cdot024\,(X/Y) \text{ in.} \qquad (89)$$

The value of x corresponds to a distance along the axis from the apex of a little more than one half of the length of the root; the value of y is small (and the centre of rotation is consequently close to the axis of the root) unless the applied force is nearly axial. This is shown by the following table :—

WEDGE-SHAPED MODEL OF THE UPPER CENTRAL TOOTH.

Inclination to axis of root of force applied at biting edge $= \tan^{-1}(Y/X)$.	Distance of centre of rotation from axis of root $= y$.
0°	∞
1°	$2\cdot9a = 1\cdot4$ in.
2°	$1\cdot4a = 0\cdot70$ in.
10°	$0\cdot28a = 0\cdot14$ in.
20°	$0\cdot14a = 0\cdot067$ in.
30°	$0\cdot087a = 0\cdot042$ in.
45°	$0\cdot050a = 0\cdot024$ in.
90°	0

We may sum up the preceding results in the following manner :

THEOREM X.—*For the wedge-shaped model of the upper central tooth, with a membrane of uniform thickness, the application of a force at the biting edge makes the tooth turn about* *a centre of rotation which lies close to the axis of the root, unless the applied force is nearly axial; the perpendicular, dropped from the centre of rotation on the axis of the root, cuts that axis at a distance from the apex slightly greater than half the length of the root.*

Let us now consider the distribution of pressure in the uniform membrane of the general wedge-shaped model. This is given in terms of the displacement of the tooth by (71). Substituting for the K's from (84), carrying out the integrations, and substituting from (86) to obtain (91), we have the following result:

THEOREM XI.—*The pressure in the uniform membrane of the wedge-shaped model is given in terms of the displacement of the tooth by the formula:*

$$p - \mathrm{P} = -6\mu u h^{-3} \tan\alpha \sec\alpha\,(a^2 - x^2) \pm 2\mu h^{-3} \sec\alpha \,.\, x(a - x)\,[3v + (a + x)\,\omega \sec^2\alpha], \quad (90)$$

where the upper sign refers to the side of the root for which $y > 0$, and the lower sign to the side for which $y < 0$: the pressure is given in terms of the applied force-system by the formula:

$$p - \mathrm{P} = -\tfrac{3}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\mathrm{X}}{\mathrm{A}} \pm \frac{12\tan\alpha}{\mathrm{A}}\,\frac{x}{a}\left(1 - \frac{x}{a}\right)\left[\left(3\mathrm{Y} - 5\,\frac{\mathrm{N}}{a}\cos^2\alpha\right) + \frac{x}{a}\left(10\,\frac{\mathrm{N}}{a}\cos^2\alpha - 5\mathrm{Y}\right)\right], \quad (91)$$

where $\mathrm{A} = 2ab\tan\alpha$, *the area of the rectangular cross-section at the margin of the membrane.*

Any applied force-system which reduces the pressure to zero at some point, or points, of the membrane, without making the pressure negative at any other point, we shall call a *critical load.*

Considering the case of a purely axial load, the pressure is given by

$$p = \mathrm{P} - \tfrac{3}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\mathrm{X}}{\mathrm{A}}, \quad (92)$$

and therefore the pressure increases or decreases steadily from the apex to the margin according as X is positive or negative. Hence we have the result:

THEOREM XII.—*The critical axial load for the wedge-shaped model is a pulling force*

$$\mathrm{X} = \tfrac{2}{3}\mathrm{AP}, \quad (93)$$

where A *is the area of the rectangular cross-section at the margin of the membrane, and* P *is the atmospheric pressure. For the wedge-shaped model of the upper central tooth, this gives*

$$\mathrm{X} = 0\cdot65 \text{ lbs.}, \quad (93\text{A})$$

when the atmospheric pressure is 15 *lbs. per sq. in.*

The following result is an immediate consequence of (91):—

THEOREM XIII.—*Under a purely transverse load* ($\mathrm{X} = 0$), *the pressure at the apex of the wedge-shaped model is equal to the atmospheric pressure.*

If, as before, we suppose that a single transverse force Y is applied at the biting edge of the upper central tooth, we are to put

$$N = \rho Y = 1\cdot 80aY\,;$$

hence the following result is a consequence of (91) :

THEOREM XIV.—*The distribution of pressure in the membrane of the wedge-shaped model of the upper central tooth, under the application of a transverse force of* Y *lbs. at the biting edge, is given in lbs. per sq. in. by the formula*

$$p = P \pm 600\ Y\frac{x}{a}\left(1 - \frac{x}{a}\right)\left(\frac{x}{a} - 0\cdot 46\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

where the dual sign is to be interpreted as in Theorem XI.

We note that the pressure is equal to the atmospheric pressure, not only at the margin of the membrane and at the apex, but also at the points for which $x = 0\cdot 46a$, a value which is, of course, independent of the values of Y and P.

There are four points of maximum and minimum pressure, namely, those points for which

$$x = 0\cdot 78a = 0\cdot 38 \text{ in.}, \qquad x = 0\cdot 20a = 0\cdot 10 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

It is easy to distinguish the maxima from the minima by the fact that at a maximum the tooth approaches the socket as a result of the displacement, which is a rotation about a point close to the middle point of the axis of the root. In fig. 8, M_1, M_3 are

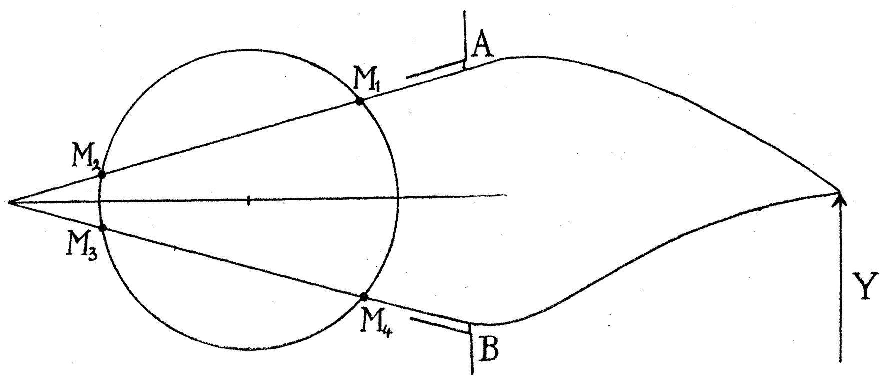


FIG. 8.

the points of maximum pressure and M_2, M_4 are the points of minimum pressure. We obtain the following values :—

Pressure at M_1 = P + 33Y lbs. per sq. in. ;
Pressure at M_2 = P − 25Y lbs. per sq. in. ;
Pressure at M_3 = P + 25Y lbs. per sq. in. ;
Pressure at M_4 = P − 33Y lbs. per sq. in.

Hence we see that the critical transverse load applied at the biting edge of the wedge-shaped model of the upper central tooth is

$$Y = P/33 = 0 \cdot 45 \text{ lbs.}, \qquad \ldots \ldots \ldots \ldots \qquad (96)$$

when the atmospheric pressure is 15 *lbs. per sq. in.*

Fig. 9 shows the pressure distribution on the two sides of the wedge-shaped model of the upper central tooth for the critical transverse force Y, applied to the biting edge. The points M_1, M_2, M_3, M_4, are those shown in fig. 8.

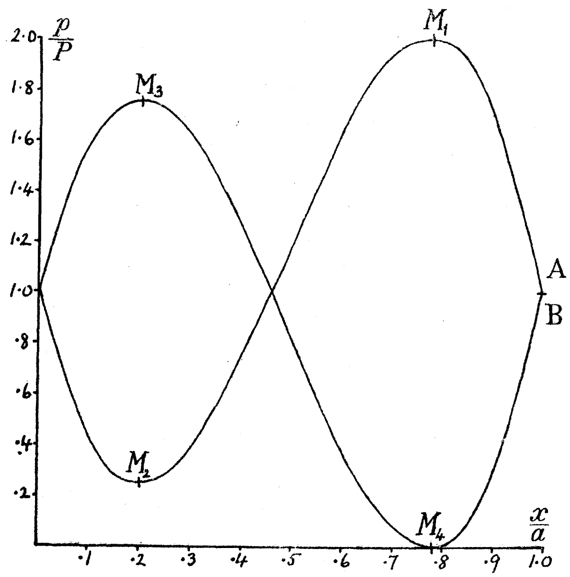
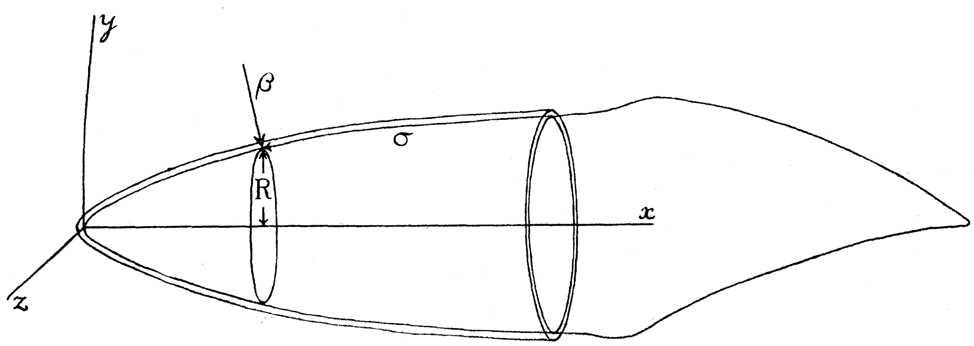


FIG. 9.

PART III.—THE PROBLEM OF THE TOOTH OF REVOLUTION.

§ 12. *Determination of the pressure in the membrane in terms of the displacement of the tooth.*

Let us now consider the case where the socket S and the root of the tooth T have surfaces of revolution about a common axis. This implies that the thickness of the membrane, considered as a function of position over the surface of the root, has the same symmetry of revolution. We shall suppose for simplicity that the margin of the membrane is a circle normal to the axis of revolution, although there is in reality a considerable deviation from this.



IG. 10.

Let us take surface co-ordinates σ, ϕ, on S (or on T), where σ is the length of the meridian measured to the general point from the margin of the membrane, and ϕ is the azimuthal angle. Equation (62) is then satisfied by the pressure p in the form

$$\frac{1}{R}\frac{\partial}{\partial\sigma}\left(Rh^3\frac{\partial p}{\partial\sigma}\right)+\frac{h^3}{R^2}\frac{\partial^2 p}{\partial\phi^2}=12\mu\beta, \quad \ldots \ldots \ldots \quad (97)$$

where R is the radius of the section $\sigma=$ constant, h is the thickness of the membrane (a function of σ only), μ is the rigidity, and β is the normal displacement of T, counted positive when away from S. The boundary condition is

$$p=P \quad \text{for} \quad \sigma=0, \quad \ldots \ldots \ldots \ldots \quad (98)$$

where P is the constant atmospheric pressure, together with the condition that p shall be single-valued for $\sigma=l$, where l is the length of the meridian from margin to apex.

In (97) β is a function of σ and ϕ, known when the infinitesimal rigid body displacement of T is known. To express it, let us introduce rectangular axes having the apex O for origin and the axis of symmetry for x-axis. The displacement of T may be resolved into infinitesimal translations u, v, w, in the directions of these axes, and infinitesimal rotations ω_1, ω_2, ω_3, about them. The displacement of a point x, y, z, of T has components

$$u-y\omega_3+z\omega_2, \quad v-z\omega_1+x\omega_3, \quad w-x\omega_2+y\omega_1, \quad \ldots \quad (99)$$

and hence we find, if ϕ is measured from the axis of y,

$$\beta=-u\frac{dR}{d\sigma}+\cos\phi\left[v\frac{dx}{d\sigma}+\omega_3\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)\right]+\sin\phi\left[w\frac{dx}{d\sigma}-\omega_2\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)\right]. \quad (100)$$

Hence we may state the following result :—

THEOREM XV.—*In the case of a tooth whose root is a surface of revolution and whose membrane has the same symmetry of revolution, the pressure in the membrane satisfies the partial differential equation*

$$\begin{aligned}\frac{1}{R}\frac{\partial}{d\sigma}\left(Rh^3\frac{\partial p}{\partial\sigma}\right)+\frac{h^3}{R^2}\frac{\partial^2 p}{\partial\phi^2} &= -12\mu u\frac{dR}{\partial\sigma}\\ &+12\mu\cos\phi\left[v\frac{dx}{d\sigma}+\omega_3\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)\right]\\ &+12\mu\sin\phi\left[w\frac{dx}{d\sigma}-\omega_2\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)\right]. \quad \ldots \quad (101)\end{aligned}$$

The component ω_1 does not appear in the above equation, since it contributes nothing to β. Except for the factors $\cos\phi$ and $\sin\phi$, the right-hand side of (101) is a known function of σ; it is therefore obvious that we should seek a solution of the form

$$p=f(\sigma)+g_1(\sigma)\cos\phi+g_2(\sigma)\sin\phi. \quad \ldots \ldots \ldots \quad (102)$$

When we substitute this in (101), we obtain the three equations

$$\frac{1}{\mathrm{R}}\frac{d}{d\sigma}\left(\mathrm{R}h^3\frac{df}{d\sigma}\right) = -12\mu u\frac{d\mathrm{R}}{d\sigma}, \quad \ldots \quad (103)$$

$$\frac{1}{\mathrm{R}}\frac{d}{d\sigma}\left(\mathrm{R}h^3\frac{dg_1}{d\sigma}\right) - \frac{h^3g_1}{\mathrm{R}^2} = 12\mu v\frac{dx}{d\sigma} + 12\mu\omega_3\left(x\frac{dx}{d\sigma} + \mathrm{R}\frac{d\mathrm{R}}{d\sigma}\right), \quad \ldots \quad (104)$$

$$\frac{1}{\mathrm{R}}\frac{d}{d\sigma}\left(\mathrm{R}h^3\frac{dg_2}{d\sigma}\right) - \frac{h^3g_2}{\mathrm{R}^2} = 12\mu w\frac{dx}{d\sigma} - 12\mu\omega_2\left(x\frac{dx}{d\sigma} + \mathrm{R}\frac{d\mathrm{R}}{d\sigma}\right). \quad \ldots \quad (105)$$

The boundary conditions for the functions f, g_1, g_2 are

$$f(0) = \mathrm{P}, \quad g_1(0) = 0, \quad g_2(0) = 0, \quad g_1(l) = 0, \quad g_2(l) = 0, \quad \ldots \quad (106)$$

the last two arising from the fact that p must be single valued at the apex.

From (103) we obtain

$$\mathrm{R}h^3\frac{df}{d\sigma} = -6\mu u\mathrm{R}^2 + \mathrm{C}, \quad \ldots \quad (107)$$

where C is a constant of integration. Hence

$$f = -6\mu u\int_0^\sigma \mathrm{R}h^{-3}\,d\sigma + \mathrm{C}\int_0^\sigma \mathrm{R}^{-1}h^{-3}\,d\sigma + \mathrm{C}', \quad \ldots \quad (108)$$

where C′ is another constant of integration. To avoid a singularity at the apex, we must take $\mathrm{C} = 0$; then (106) gives $\mathrm{C}' = \mathrm{P}$. Hence f is given by

$$f = \mathrm{P} - 6\mu u\int_0^\sigma \mathrm{R}h^{-3}\,d\sigma. \quad \ldots \quad (109)$$

We may state the following result :—

THEOREM XVI.—*In the case of a tooth of revolution the pressure at the apex is*

$$(p)_0 = \mathrm{P} - 6\mu u\int_0^l \mathrm{R}h^{-3}\,d\sigma\,; \quad \ldots \quad (110)$$

it is independent of all the components of the displacement of the tooth except the axial translation; if the axial translation is zero, the pressure at the apex is equal to the atmospheric pressure.

Let us now project on the plane through the apex normal to the axis of symmetry, the vectors representing respectively the translation u, v, w, and the rotation $\omega_1, \omega_2, \omega_3$. The two projections will be perpendicular to one another if

$$v\omega_2 + w\omega_3 = 0, \quad \text{or} \quad v/w = -\omega_3/\omega_2. \quad \ldots \quad (111)$$

If this condition is satisfied, the ratio of the right-hand sides of (104) and (105) is a constant, and hence (on account of the boundary conditions (106)) the ratio $g_1 : g_2$ will be a constant, namely, the ratio $v : w$. We may state the following result :—

THEOREM XVII.—*When the displacement of the tooth of revolution is such that the components of the vectors of translation and rotation in the plane normal to the axis of symmetry are perpendicular to one another, then along the section of the membrane by the plane through the axis of symmetry which contains the axis of rotation, the pressure p depends only on the axial translation, being given by*

$$p = \mathrm{P} - 6\mu u \int_0^\sigma \mathrm{R}h^{-3}\, d\sigma. \qquad (112)$$

This follows from (102) and (109), since for the section considered,

$$\tan\phi = -v/w = -g_1/g_2.$$

The above theorem is of especial interest in the case of a two-dimensional displacement of the tooth, to be discussed more fully later.

Let us now proceed with the integration of (104) and (105), without the above special assumption concerning the displacement, *but assuming that the thickness* (h) *of the membrane is constant.* Introducing a new independent variable

$$\zeta = \int_0^\sigma \frac{d\sigma}{\mathrm{R}}, \qquad (113)$$

so that $\zeta = 0$ when $\sigma = 0$, and $\zeta = \infty$ when $\sigma = l$, the equations (104) and (105) transform into

$$(d^2 g_1/d\zeta^2) - g_1 = \mathrm{F}_1(\zeta), \qquad (114)$$

and

$$(d^2 g_2/d\zeta^2) - g_2 = \mathrm{F}_2(\zeta), \qquad (115)$$

respectively, where we have written for abbreviation

$$\mathrm{F}_1(\zeta) = \frac{12\mu v}{h^3}\mathrm{R}\frac{dx}{d\zeta} + \frac{12\mu\omega_3}{h^3}\mathrm{R}\left(x\frac{dx}{d\zeta} + \mathrm{R}\frac{d\mathrm{R}}{d\zeta}\right), \qquad (116)$$

$$\mathrm{F}_2(\zeta) = \frac{12\mu w}{h^3}\mathrm{R}\frac{dx}{d\zeta} - \frac{12\mu\omega_2}{h^3}\mathrm{R}\left(x\frac{dx}{d\zeta} + \mathrm{R}\frac{d\mathrm{R}}{d\zeta}\right). \qquad (117)$$

The differential equations (114), (115) for g_1, g_2 have the general solutions

$$g_1 = \mathrm{C}_1 e^{\zeta} + \mathrm{C}'_1 e^{-\zeta} + \tfrac{1}{2}e^{\zeta}\int_0^\zeta e^{-\lambda}\mathrm{F}_1(\lambda)\, d\lambda - \tfrac{1}{2}e^{-\zeta}\int_0^\zeta e^{\lambda}\mathrm{F}_1(\lambda)\, d\lambda, \qquad (118)$$

$$g_2 = \mathrm{C}_2 e^{\zeta} + \mathrm{C}'_2 e^{-\zeta} + \tfrac{1}{2}e^{\zeta}\int_0^\zeta e^{-\lambda}\mathrm{F}_2(\lambda)\, d\lambda - \tfrac{1}{2}e^{-\zeta}\int_0^\zeta e^{\lambda}\mathrm{F}_2(\lambda)\, d\lambda, \qquad (119)$$

where C_1, C'_1, C_2, C'_2 are arbitrary constants.

Let us consider the behaviour of these expressions as ζ tends to infinity, *i.e.*, as we approach the apex of the root, where $\mathrm{R} = 0$. Assuming that the curve whose revolution generates the surface of the root is analytic and does not touch the axis of symmetry at the apex, we have near the apex

$$x = a_1 \mathrm{R} + a_2 \mathrm{R}^2 + \dots , \qquad (120)$$

and hence along a meridian

$$d\sigma^2 = dx^2 + d\mathrm{R}^2 = d\mathrm{R}^2 [1 + (a_1 + 2a_2 \mathrm{R} + \dots)^2], \qquad (121)$$

so that

$$d\zeta = \frac{d\sigma}{\mathrm{R}} = -\frac{d\mathrm{R}}{\mathrm{R}} [1 + (a_1 + 2a_2 \mathrm{R} + \dots)^2]^{\frac{1}{2}}. \qquad (122)$$

Therefore, for large values of ζ, R is of the order of $e^{-\zeta}$. Now

$$\mathrm{F}_1(\zeta) = -\left[\frac{12\mu v}{h^3} \mathrm{R} \frac{dx}{d\mathrm{R}} + \frac{12\mu\,\omega_3}{h^3} \mathrm{R}\left(x \frac{dx}{d\mathrm{R}} + \mathrm{R}\right)\right] \frac{\mathrm{R}}{[1 + (a_1 + 2a_2 \mathrm{R} + \dots)^2]^{\frac{1}{2}}}, \qquad (123)$$

and thus for large values of ζ, F_1 is of the order of R^2 or $e^{-2\zeta}$. Hence it is evident that

$$\int_0^\infty e^{-\lambda} \mathrm{F}_1(\lambda)\, d\lambda, \qquad \int_0^\infty e^{\lambda} \mathrm{F}_1(\lambda)\, d\lambda$$

exist, as also do the same integrals with F_2 instead of F_1.

The boundary conditions (106) require that the constants in (118), (119) satisfy the equations

$$\mathrm{C}_1 + \mathrm{C}'_1 = 0, \qquad \mathrm{C}_2 + \mathrm{C}'_2 = 0, \qquad (124)$$

and hence it is easy to see that the functions g_1, g_2, satisfying respectively (104), (105) and also the boundary conditions (106), may be written

$$g_1 = -\sinh \zeta \int_\zeta^\infty e^{-\lambda} \mathrm{F}_1(\lambda)\, d\lambda - e^{-\zeta} \int_0^\zeta \sinh\lambda\, \mathrm{F}_1(\lambda)\, d\lambda, \qquad (125)$$

$$g_2 = -\sinh \zeta \int_\zeta^\infty e^{-\lambda} \mathrm{F}_2(\lambda)\, d\lambda - e^{-\zeta} \int_0^\zeta \sinh\lambda\, \mathrm{F}_2(\lambda)\, d\lambda. \qquad (126)$$

Let us now state our result.

THEOREM XVIII.—*In the case of a tooth of revolution with a membrane of uniform thickness h, the pressure p in the membrane is expressed in terms of the rigid body displacement of the tooth by the formula*

$$p = \mathrm{P} - 6\mu u h^{-3} \int_0^\sigma \mathrm{R}\, d\sigma + g_1 \cos\phi + g_2 \sin\phi, \qquad (127)$$

where P *is the atmospheric pressure,* μ *is the rigidity of the membrane,* u *is the axial translation,* R *is the distance from the axis of symmetry,* σ *is the meridian distance from the margin of the membrane,* ϕ *is the azimuthal angle measured from the plane of xy, and* g_1, g_2 *are functions of* ζ *and the components* v, w, ω_2, ω_3 *of the displacement of the tooth as given in* (125), (126), ζ *being defined by* (113) *and* F_1, F_2 *by* (116), (117).

§13. *The case of a plane displacement: connection between the applied force-system and the displacement of the tooth.*

In the discussion of the preceding section the displacement of the tooth was general. We shall now suppose that this displacement is a plane (or two-dimensional) displacement in a meridian plane of the root. Let us choose our axes so that $z = 0$ is the plane of the displacement. The displacement may be resolved into an axial translation u, a transverse translation v, and a rotation ω_3 about an axis through the apex perpendicular to the axis of symmetry and to the transverse translation. The preceding theory applies, but now, since $w = 0$, $\omega_2 = 0$, we have, by (117), $F_2 = 0$, and therefore, by (126), $g_2 = 0$. Accordingly, the general expression (127) for the pressure p is simplified by the vanishing of the last term, so that, *in the case of a plane displacement, the pressure at any point of the membrane (of uniform thickness h) is given by*

$$p = P - 6\mu u h^{-3} \int_0^\sigma R \, d\sigma + g_1 \cos \phi. \quad \ldots \ldots \quad (128)$$

Along the meridian section perpendicular to the plane of the displacement (*i.e.*, corresponding to $\phi = \pm \frac{1}{2}\pi$) the pressure is given by the first two terms of this expression.

It is evident that the maximum and minimum values of p occur in the section of the membrane by the plane of the displacement ($\phi = 0$ or π); using the expression (125) for g_1, we see that *the points of maximum and minimum pressure may be determined by the equations*

$$\left.\begin{array}{c} \phi = 0 \text{ or } \pi, \\ 6\mu u h^{-3} R^2 \pm \cosh \zeta \int_\zeta^\infty e^{-\lambda} F_1(\lambda)\, d\lambda \mp e^{-\zeta} \int_0^\zeta \sinh \lambda \, F_1(\lambda)\, d\lambda = 0, \end{array}\right\} \quad . \quad (129)$$

where the upper signs apply when $\phi = 0$ and the lower signs when $\phi = \pi$.

To determine which are maxima and which are minima, we refer to (97), which, since $\partial p / \partial \sigma = 0$ at such points, reads

$$\frac{\partial^2 p}{\partial \sigma^2} + \frac{1}{R^2} \frac{\partial^2 p}{\partial \phi^2} = \frac{12 \mu \beta}{h^3}. \quad \ldots \ldots \quad (130)$$

*It is obvious that at a maximum of p, β is negative (i.e., the tooth approaches the socket as the result of the displacement), and that at a minimum of p, β is positive (i.e., the tooth recedes from the socket).**

Let us now consider the applied force-system which produces the plane displacement u, v, ω_3. By symmetry, it can be reduced to forces X, Y acting at the apex O along

* It is easily seen from (59) that this method of distinguishing between maxima and minima is valid also in the general case, where the root is not necessarily of revolution nor the membrane of uniform thickness.

the axes of x and y respectively, together with a couple N whose axis is the axis of z. We assume that the part of the tooth which is not covered with membrane is subjected to the uniform atmospheric pressure P, and we do not count the resultant of this pressure as part of the applied force-system.

The conditions of statical equilibrium of the tooth give the equations

$$\left.\begin{aligned} X &= \int_{\sigma=0}^{\sigma=l}\int_{\phi=0}^{\phi=2\pi}(p-P)\,R\,\frac{dR}{d\sigma}\,d\sigma\,d\phi, \\ Y &= -\int_{\sigma=0}^{\sigma=l}\int_{\phi=0}^{\phi=2\pi}(p-P)\,R\cos\phi\,\frac{dx}{d\sigma}\,d\sigma\,d\phi, \\ N &= -\int_{\sigma=0}^{\sigma=l}\int_{\phi=0}^{\phi=2\pi}(p-P)\,R\cos\phi\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)d\sigma\,d\phi. \end{aligned}\right\} \quad (131)$$

Inserting the value of p from (128), we obtain the following result :—

THEOREM XIX.—*In the case of a tooth of revolution, with a membrane of uniform thickness h, the following equations connect the applied force-system and the displacement of the tooth, when this displacement is a plane displacement :*

$$\left.\begin{aligned} X &= 6\pi\mu u h^{-3}\int_0^l R^3\,d\sigma, \\ Y &= -\pi\int_0^l g_1 R\,\frac{dx}{d\sigma}\,d\sigma, \\ N &= -\pi\int_0^l g_1 R\left(x\frac{dx}{d\sigma}+R\frac{dR}{d\sigma}\right)d\sigma, \end{aligned}\right\} \quad (132)$$

where g_1 involves the components v and ω_3 of the displacement of the tooth, being given by (125) *and* (116).

It may be remarked that if h^{-3} is brought under the sign of integration in the first of these equations, they are valid for the more general case in which h is not a constant, but is a function of σ only; for this more general case, g_1 has not been evaluated explicitly; it is to be determined from the differential equation (104), with the boundary conditions (106).

§ 14. *The conical model.*

Let us now consider a model tooth whose root is a right circular cone. This is the simplest and most convenient geometrical figure which we can choose to represent the average shape of the root of the upper central tooth. The approximation is not too bad, although in reality the cross-section of the root deviates from the circular shape in the direction of the triangular. For simplicity, we shall confine ourselves to the case of a membrane of uniform thickness h, although in the human tooth the membrane varies considerably in thickness, being thickest at the margin and the apex, and thinnest about half way between. We shall confine our attention to the plane displacement discussed in § 13, which is the type of displacement caused by the application to the tooth of a single force whose line of action intersects the axis of the conical root.

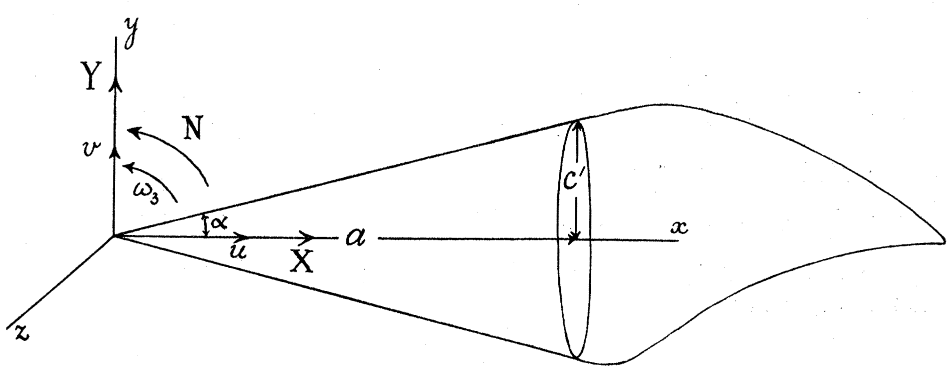


Fig. 11.

We shall choose axes as in § 13, the origin being at the apex of the root and the axis of x being the axis of symmetry. The axis of z will be perpendicular to the plane of the displacement. We shall adopt the following notation, the numerical values being those which we shall employ in making numerical calculations for the upper central tooth (*cf.* § 1):

$$\left.\begin{array}{l} \text{Length of root} = a = 0\cdot 49 \text{ in.} \\ \text{Radius of cross-section of root at margin of membrane (supposed circular)} \\ \quad = c' = 0\cdot 127 \text{ in.*} \\ \text{Semi-angle of root} = \alpha = \tan^{-1}(c'/a) = 14^\circ\ 35' \\ \text{Distance of biting edge from apex} = \rho = 0\cdot 88 \text{ in.} \\ \text{Area of cross-section of root at margin of membrane} = \mathrm{A} = \pi c'^2 = \\ \quad 0\cdot 051 \text{ sq. in.} \end{array}\right\} \quad (133)$$

We shall first develop the relations (132) between the applied force-system and the components of the displacement of the tooth.

From the geometrical shape of the root we have

$$\left.\begin{array}{ll} \mathrm{R} = x\tan\alpha, & \sigma = (a - x)\sec\alpha, \\ \zeta = \int_0^\sigma \mathrm{R}^{-1}\,d\sigma = \operatorname{cosec}\alpha\log(a/x), & x = ae^{-\zeta\sin\alpha}. \end{array}\right\} \quad . \quad . \quad . \quad (134)$$

From the first of (132) *the axial component of force is given in terms of the axial translation by*

$$\mathrm{X} = \tfrac{3}{2}\pi\mu u a^4 h^{-3}\sec\alpha\tan^3\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

By (116) we have

$$\mathrm{F}_1(\zeta) = -12\mu a^2 h^{-3}\sin\alpha\tan\alpha\,(ve^{-2\zeta\sin\alpha} + a\omega_3\sec^2\alpha\, e^{-3\zeta\sin\alpha}), \quad . \quad . \quad (136)$$

and therefore by (125)

$$g_1 = 12\mu a^2 h^{-3}\sin\alpha\tan\alpha\left(v\frac{e^{-2\zeta\sin\alpha} - e^{-\zeta}}{1 - 4\sin^2\alpha} + a\omega_3\sec^2\alpha\,\frac{e^{-3\zeta\sin\alpha} - e^{-\zeta}}{1 - 9\sin^2\alpha}\right). \quad (137)$$

* This value is half the arithmetic mean of the diameters given in § 1.

Hence by (132) we have

$$\left.\begin{aligned} Y &= 12\pi\,\mu a^4 h^{-3}\,\frac{\cos\alpha\tan^3\alpha}{1+2\sin\alpha}\left(\tfrac{1}{4}\,v\,\frac{1}{1+2\sin\alpha}+\tfrac{1}{5}a\,\omega_3\sec^2\alpha\,\frac{1}{1+3\sin\alpha}\right),\\ N &= 12\pi\,\mu a^5 h^{-3}\,\frac{\sec\alpha\tan^3\alpha}{1+3\sin\alpha}\left(\tfrac{1}{5}\,v\,\frac{1}{1+2\sin\alpha}+\tfrac{1}{6}a\,\omega_3\sec^2\alpha\,\frac{1}{1+3\sin\alpha}\right).\end{aligned}\right\} \quad (138)$$

Hence we may state the result :

THEOREM XX.—*Equations* (135) *and* (138) *express the applied force-system in terms of the plane displacement of the conical model. The displacement is therefore given in terms of the force-system by*

$$\left.\begin{aligned} u &= \frac{2h^3\cos\alpha\cot^3\alpha}{3\pi\,\mu a^4}\,X,\\ v &= \frac{5h^3\cot^3\alpha\,(1+2\sin\alpha)}{3\pi\,\mu a^5}\left[5aY\sec\alpha\,(1+2\sin\alpha)-6N\cos\alpha\,(1+3\sin\alpha)\right],\\ \omega_3 &= -\,\frac{5h^3\cos^2\alpha\cot^3\alpha\,(1+3\sin\alpha)}{2\pi\,\mu a^6}\left[4aY\sec\alpha\,(1+2\sin\alpha)\right.\\ &\qquad\qquad\left.-\,5N\cos\alpha\,(1+3\sin\alpha)\right],\end{aligned}\right\} \quad (139)$$

in which h is the uniform thickness of the membrane, μ *is its rigidity, a is the length of the root, and* α *is the semi-angle of the root.*

Since the displacement of the tooth is two-dimensional, it is equivalent to a rotation about an axis perpendicular to the plane of the displacement. We shall call the point where the axis of rotation cuts the diametral plane of the displacement, *the centre of rotation ;* its co-ordinates are

$$x = -\,v/\omega_3 \quad , \quad y = u/\omega_3 \quad , \quad z = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (140)$$

Let us suppose that the tooth is subjected to a single force Y, whose line of action intersects the axis of symmetry at the point $(\rho, 0, 0)$. We have then $N = \rho\,Y$, and the position of the centre of rotation is given by

$$\left.\begin{aligned} x &= \tfrac{2}{3}a\sec^2\alpha\,\frac{1+2\sin\alpha}{1+3\sin\alpha}\,\frac{6\,(\rho/a)\cos^2\alpha\,(1+3\sin\alpha)-5\,(1+2\sin\alpha)}{5\,(\rho/a)\cos^2\alpha\,(1+3\sin\alpha)-4\,(1+2\sin\alpha)},\\ y &= \tfrac{4}{15}a\,\frac{1}{1+3\sin\alpha}\,\frac{1}{5\,(\rho/a)\cos^2\alpha\,(1+3\sin\alpha)-4\,(1+2\sin\alpha)}\,\frac{X}{Y}.\end{aligned}\right\} \quad (141)$$

Just as with the wedge-shaped model, for a given conical tooth the value of x depends only on the distance from the apex to the point where the line of action of the applied force cuts the axis of the root, whereas the value of y depends, not only on this distance, but also on the inclination of the applied force to the axis of the root.

Let us now insert the numerical values for the upper central tooth ; we find that *the centre of rotation of the conical model of the upper central tooth, under the action of a force applied at the biting edge, is situated at the point whose co-ordinates are*

$$x = 0{\cdot}72\,a = 0{\cdot}35\text{ in.}, \quad y = 0{\cdot}17\,a\,(X/Y) = 0{\cdot}0085\,(X/Y)\text{ in.} \quad . \; . \quad (142)$$

This result differs quite considerably from the corresponding result for the wedge-shaped model (*cf.* (89)). We see that for the conical model the value of x corresponds to a distance from the apex slightly less than three-quarters of the length of the root. As with the wedge-shaped model, the distance of the centre of rotation from the axis of the root is small, unless the applied force is nearly axial; this is shown in the following table :—

CONICAL MODEL OF THE UPPER CENTRAL TOOTH.

Inclination to axis of root of force applied at biting edge $= \tan^{-1}(Y/X)$.	Distance of centre of rotation from axis of root $= y$.
0°	∞
1°	$0 \cdot 99a = 0 \cdot 49$ in.
2°	$0 \cdot 50a = 0 \cdot 24$ in.
10°	$0 \cdot 098a = 0 \cdot 048$ in.
20°	$0 \cdot 048a = 0 \cdot 023$ in.
30°	$0 \cdot 030a = 0 \cdot 015$ in.
45°	$0 \cdot 017a = 0 \cdot 008$ in.
90°	0

Summing up, we may state the following result :—

THEOREM XXI.—*For the conical model of the upper central tooth with a membrane of uniform thickness, the application of a force at the biting edge makes the tooth turn about an axis which passes close to the axis of the root, unless the force is nearly axial; the common perpendicular to the axis of the root and axis of rotation meets the former at a distance from the apex slightly less than three-quarters of the length of the root.*

Let us now consider the distribution of pressure in the membrane. This is given by (128), in which

$$\int_0^\sigma R\,d\sigma = \tfrac{1}{2}\sec\alpha\tan\alpha\,(a^2 - x^2), \quad \text{. . . . . (143)}$$

and g_1 is given by (137); hence we may state the result :

THEOREM XXII.—*The pressure in the membrane of the conical model* (*the membrane being of uniform thickness* h) *is given in terms of the displacement of the tooth by the formula*

$$p - P = -3\mu u h^{-3}\sec\alpha\tan\alpha\,(a^2 - x^2)$$

$$+ 12\mu a^2 h^{-3}\sin\alpha\tan\alpha\cos\phi\left[\frac{v}{1 - 4\sin^2\alpha}\left\{\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^{\operatorname{cosec}\alpha}\right\}\right.$$

$$\left. + \frac{a\,\omega_3\sec^2\alpha}{1 - 9\sin^2\alpha}\left\{\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right)^{\operatorname{cosec}\alpha}\right\}\right]. \quad (144)$$

It is convenient to regard this excess over atmospheric pressure as due to (i) the axial displacement u; (ii) the transverse displacement v, ω_3, or, equivalently, to (i) the axial load X; (ii) the transverse load Y, N. Thus we may state the result:

THEOREM XXIII.—*In the case of the conical model with a membrane of uniform thickness, the excess of the pressure in the membrane over atmospheric pressure, due to the axial load* X, *is, by* (139),

$$p - \mathrm{P} = -\frac{2\mathrm{X}}{\mathrm{A}}\left(1 - \frac{x^2}{a^2}\right), \qquad (145)$$

where A is the area of the cross-section of the root at the margin of the membrane, and a is the length of the root; the excess of pressure, due to the transverse load Y, N, *is*

$$p - \mathrm{P} = \frac{10\tan\alpha\cos\phi}{\mathrm{A}}\left[\frac{2}{1-2\sin\alpha}\left\{\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^{\operatorname{cosec}\alpha}\right\}\left\{5\mathrm{Y}(1+2\sin\alpha) - 6\frac{\mathrm{N}}{a}\cos^2\alpha(1+3\sin\alpha)\right\}\right.$$
$$\left. - \frac{3}{1-3\sin\alpha}\left\{\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right)^{\operatorname{cosec}\alpha}\right\}\left\{4\mathrm{Y}(1+2\sin\alpha) - 5\frac{\mathrm{N}}{a}\cos^2\alpha(1+3\sin\alpha)\right\}\right]. \qquad (146)$$

The excess of pressure due to the total load is found by adding the right-hand sides of (145) *and* (146).

We note the following facts:

THEOREM XXIV.—*Under a purely axial load* X, *the pressure in the membrane increases or decreases steadily from* P — 2X/A *at the apex to* P *at the margin, according as* X *is positive or negative. The critical axial load* (*which reduces the pressure at the apex to zero*) *is a pulling force*

$$\mathrm{X} = \tfrac{1}{2}\mathrm{AP}. \qquad (147)$$

For the conical model of the upper central tooth, this gives

$$\mathrm{X} = 0\cdot38 \text{ lbs.}, \qquad (147\text{A})$$

when the atmospheric pressure is 15 *lbs. per sq. in.**

The above force, when applied as an axial pressure, raises the pressure at the apex to two atmospheres.

The following results are immediate consequences of (146):—

THEOREM XXV.—*The pressure in the section of the membrane by the diametral plane perpendicular to the plane of the displacement is unaffected by the transverse force-system; it is given by* (145).

THEOREM XXVI.—*Under a purely transverse load, the pressure at the apex is equal to atmospheric pressure.*†

Let us now consider in more detail the distribution of pressure due to the transverse load, as given by (146). Let us insert numerical values for the upper central tooth,

* It is interesting to compare the above values with the corresponding values for the case of the wedge-shaped model, given in (93) and (93A).

† It is easily seen, by reference to (128) and (132), that this result is true, not only for the conical model, but also for the general case of a surface of revolution.

and assume that the load consists of a single force Y, applied at the biting edge in a direction perpendicular to the axis of the root; we are therefore to put

$$N = \rho\, Y = 1 \cdot 80 a Y.$$

The data (133) give

$$\operatorname{cosec} \alpha = \operatorname{cosec} 14^\circ\, 35' = 3 \cdot 97.$$

Hence we may state the result:

THEOREM XXVII.—*The distribution of pressure in the membrane (of uniform thickness) of the conical model of the upper central tooth, under the application of a transverse force* Y *lbs. at the biting edge, is given in lbs. per sq. in. by the formula*

$$p = P - 3350 Y \cos \phi\, (x/a)^2 \left[(x/a)^{1 \cdot 97} - 1 \cdot 627\, (x/a) + 0 \cdot 627 \right]. \quad . \quad . \quad (148)$$

The pressure is equal to atmospheric pressure, not only at the apex and at the margin of the membrane, but also on an intermediate circular section, given by

$$x = 0 \cdot 64 a = 0 \cdot 31 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

Let us now consider the critical transverse load Y applied at the biting edge, *i.e.*, that load which reduces the pressure to zero at some point of the membrane.

The maximum and minimum values of p, for a given load Y, occur at the points given by

$$\left. \begin{array}{c} \phi = 0 \text{ or } \pi, \\ (x/a) \left[3 \cdot 97\, (x/a)^{1 \cdot 97} - 4 \cdot 88\, (x/a) + 1 \cdot 254 \right] = 0. \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (150)$$

Leaving out of consideration the apex $x = 0$, at which we know that p is equal to the atmospheric pressure P, we find that the zeros of the remaining [ ], which is nearly quadratic, are

$$x = 0 \cdot 87 a = 0 \cdot 43 \text{ in.}, \qquad x = 0 \cdot 37 a = 0 \cdot 18 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad (151)$$

Hence the four points of stationary pressure are as follows, with the corresponding pressures in lbs. per sq. in., Y being measured in lbs. :—

$$\begin{array}{llll} M_1 \ldots \phi = 0, & x = 0 \cdot 87 a, & p = P + 79 Y, \\ M_2 \ldots \phi = 0, & x = 0 \cdot 37 a, & p = P - 76 Y, \\ M_3 \ldots \phi = \pi, & x = 0 \cdot 37 a, & p = P + 76 Y, \\ M_4 \ldots \phi = \pi, & x = 0 \cdot 87 a, & p = P - 79 Y. \end{array}$$

Let us (without loss of generality) assume Y to be positive. The highest pressure occurs at M_1 and the lowest at M_4. We may state the following result:—

THEOREM XXVIII.—*When the conical model of the upper central tooth, with a membrane of uniform thickness, is subjected to a positive transverse force* Y *at the biting edge, the points of maximum and minimum pressure are the points* M_1, M_2, M_3, M_4, *shown in*

Fig. 12, *and described above. The greatest and least pressures occur at* M_1 *and* M_4 *respectively. The critical transverse load is*

$$Y = P/79 \text{ lbs.}, \quad \ldots \ldots \ldots \ldots \quad (152)$$

when P *is expressed in lbs. per sq. in.; under this load, the pressures are as follows:—*

$$\left.\begin{array}{ll} \text{at } M_1,\ p = 2P\,; & \text{at } M_2,\ p = 0\cdot04P\,; \\ \text{at } M_3,\ p = 1\cdot96P\,; & \text{at } M_4,\ p = 0. \end{array}\right\} \quad \ldots \ldots \quad (153)$$

When P = 15 *lbs. per sq. in., the critical transverse load is*

$$Y = 0\cdot19 \text{ lbs.} = 3\cdot0 \text{ oz.} \quad \ldots \ldots \ldots \ldots \quad (152\text{A})$$

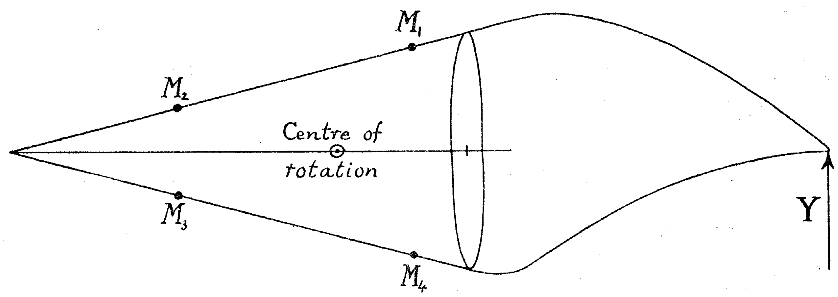


Fig. 12.

Fig. 13 shows the distribution of pressure along the generators $\phi = 0$ and $\phi = \pi$, when the tooth is subjected to the critical transverse load; the values of p/P are given by (148), on putting $Y = P/79$, and substituting the appropriate values for ϕ. The distribution of pressure may be compared with that shown in fig. 9 for the case of the wedge-shaped model.

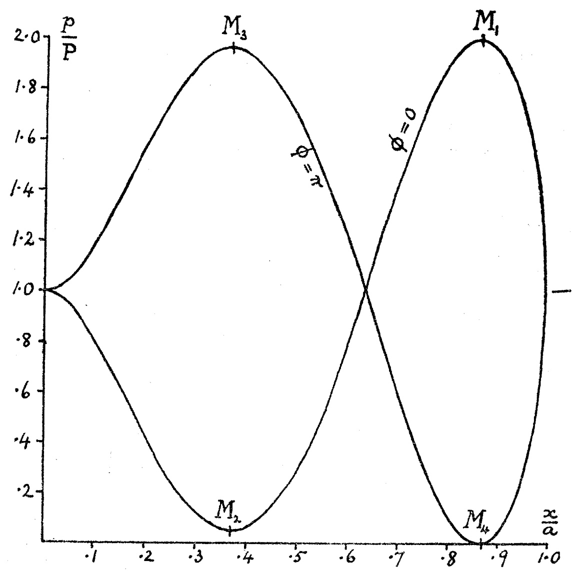


Fig. 13.

Fig. 14 shows the lines of constant pressure (or, more precisely, their projections on the plane of the displacement) when the tooth is subjected to the critical transverse load, in the sense indicated by the arrow marked Y on the right-hand side. The curves are plotted from the formula (148).

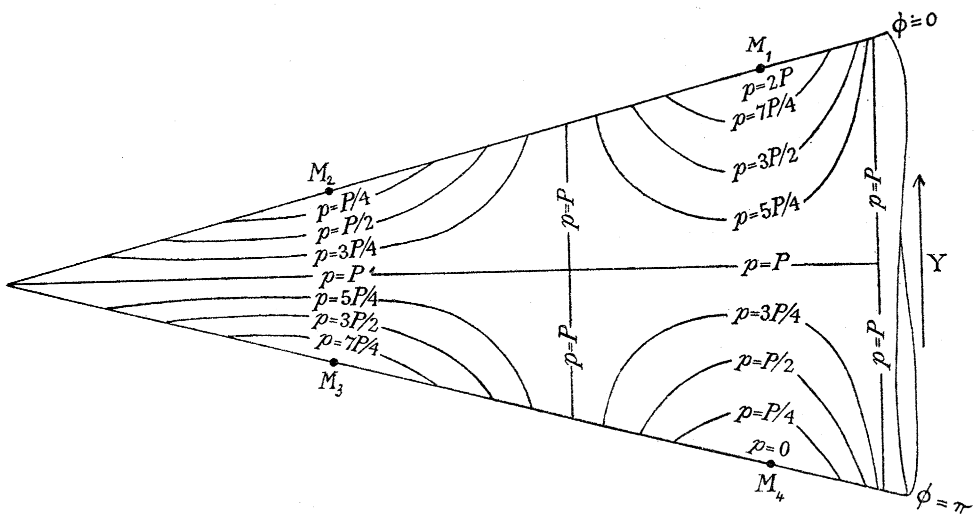


FIG. 14.

Let us return to equations (139), which connect the displacement of the conical model with the forces applied to it. We have, in (133), given numerical values for the constants a and α, and an average value for h may be taken to be (see Introduction)—

$$h = 0\cdot0095 \text{ in.}; \qquad (154)$$

but before the equations can be applied to find the numerical value of the displacement corresponding to an assigned load, we must know the numerical value of the rigidity μ of the membrane. For this no figures appear to be available, but it is interesting to see what displacements correspond to the critical axial and transverse loads if we assume that the rigidity of the membrane is the same as that of rubber, viz. :—

$$\mu = 1\cdot6 \times 10^7 \text{ dynes per sq. cm.} = 230 \text{ lbs. per sq. in.} \qquad (155)$$

Before inserting this value (which is, of course, purely tentative), let us note that by (139) the axial displacement u is given in terms of the axial load X by the equation

$$u = K_1 \frac{Xh^3}{\mu a^4}, \qquad (156)$$

where K_1 is a constant without dimensions,

$$K_1 = \frac{2 \cos \alpha \cot^3\alpha}{3\pi} = 11\cdot7, \qquad (157)$$

and that the transverse displacement v of the apex and the rotation ω_3 for a transverse load Y at the biting edge (so that $N = \rho Y$) are given by

$$v = K_2 \frac{Yh^3}{\mu a^4}, \qquad \omega_3 = K_3 \frac{Yh^3}{\mu a^5}, \quad \ldots \quad (158)$$

where K_2 and K_3 are constants without dimensions,

$$K_2 = \frac{5 \cot^3 \alpha\, (1 + 2 \sin \alpha)}{3\pi} [5 \sec \alpha\, (1 + 2 \sin \alpha) - 6\, (\rho/a) \cos \alpha\, (1 + 3 \sin \alpha)] = -480, \quad (159)$$

$$K_3 = -\frac{5 \cos^2 \alpha \cot^3 \alpha\, (1 + 3 \sin \alpha)}{2\pi} [4 \sec \alpha\, (1 + 2 \sin \alpha) - 5\, (\rho/a) \cos \alpha\, (1 + 3 \sin \alpha)] = 670\,; \quad (160)$$

the transverse displacement of the biting edge is

$$\delta = v + \rho\, \omega_3 = K_4 \frac{Yh^3}{\mu a^4}, \quad \ldots \quad (161)$$

where

$$K_4 = K_2 + (\rho/a)\, K_3 = 730. \quad \ldots \quad (162)$$

Let us now insert for h and μ the numerical values (154), (155), and for a the numerical value given in (133). If in (156) we make X equal to the critical axial load of 0·38 lbs., we find that *the axial displacement of the conical model of the upper central tooth under the influence of the critical axial load of* 0·38 *lbs. is* $2 \cdot 8 \times 10^{-7}$ *in., if the membrane has the same rigidity as rubber.* The axial displacement is proportional to the axial load; it would require (on the above assumptions) an axial load of over 1,300 lbs. to give the tooth an axial displacement of one thousandth of an inch, if the membrane were able to stand the stress, which, of course, it would not be able to do. It appears therefore that our theory explains very successfully the tightness of the teeth with respect to axial loading. It is difficult to see how an elastic cord theory could give an adequate explanation.

Let us now insert in (161) numerical values, giving to Y the value of the critical transverse load, namely, 0·19 lbs. We find that *the transverse displacement of the biting edge of the conical model of the upper central tooth, under the influence of the critical transverse load of* 0·19 *lbs. applied at the biting edge, is* $8 \cdot 9 \times 10^{-6}$ *in., if the membrane has the rigidity of rubber.* It would require a transverse load of about 21 lbs. to produce a transverse displacement of one thousandth of an inch. Here again we see that the tightness of the tooth is well explained.

It may well be that the rigidity of the membrane is considerably less than that of rubber. Its value might be determined experimentally by measuring the displacement of the edge of the tooth corresponding to a measured transverse load, and using the formula (161), the constant K_4 having perhaps a somewhat different numerical value for the tooth employed in the experiment.

The explanation of the loosening of a tooth is to be found in one or both of two phenomena. The first is an increase in the thickness of the membrane h (to the cube of which the displacement is proportional), and the second is a decrease in the rigidity μ (to which the displacement is reciprocally proportional). The latter will undoubtedly occur when the fibre-bundles break down, and are replaced by loose connective tissue, observed microscopically in the membranes of functionless teeth. The limiting case of this process of breaking down might be regarded as that in which the membrane ceases to act as an elastic solid, and becomes viscous liquid; a sustained load on the tooth will then cause the membrane to flow like a liquid, and the tooth will be brought into contact with the bone.

[*Added in proof*, *February* 10, 1933.—Since this paper was written, I have investigated the case of a tooth having a general conical root, not of revolution, and, in particular, the case where the section of the root is an equilateral triangle: these results are appearing in the 'Philosophical Magazine.']

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